INDEPENDENT SCHOOLS

2009
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Write your student number and/or name at the top of every page

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

TABLE OF STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1};\quad n \neq -1$$

$$\int \frac{1}{x} dx = \ln x,\quad x>0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note: $\ln x = \log_e x,\quad x>0$

Marks

Question 1 **Begin a new booklet**

(a) Find $\int \frac{(x+1)^2}{x}\,dx$. **2**

(b)(i) Find constants A, B, C and D such that $\frac{x^3+2x^2+4x+2}{(x^2+1)(x^2+4)} \equiv \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$. **2**

(ii) Hence evaluate $\int_0^2 \frac{x^3+2x^2+4x+2}{(x^2+1)(x^2+4)}\,dx$. **2**

(c) Use the substitution $t = \tan\frac{x}{2}$ to find $\int \frac{1}{5+4\cos x+3\sin x}\,dx$. **3**

(d) Use the substitution $u = \sin x$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx$. **3**

(e) Use the substitution $u = -x$ to evaluate $\int_{-1}^{1} \frac{1}{e^x+1}\,dx$. **3**

Student name / number ....

Marks

Question 2 **Begin a new booklet**

(a) If $z_1 = 3i$ and $z_2 = 1+i$, find the values of

(i) $|z_1 - z_2|$. **1**

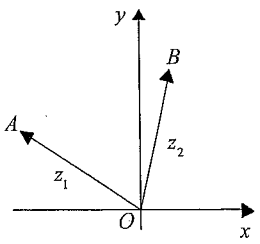
(ii) $z_1 + \overline{z}_2$. **1**

(iii) $\frac{z_1}{z_2}$. **1**

(b)(i) If $z = 1 + i\sqrt{3}$, express z, z^2 and $\frac{1}{z}$ in modulus-argument form. **3**

(ii) If the points A and B represent the complex numbers z^2 and $\frac{1}{z}$ in the Argand diagram, show that A, O and B are collinear, where O is the origin. **1**

(c)



In the Argand diagram, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers $z_1 = 2(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5})$ and $z_2 = 2(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15})$ respectively.

(i) Show that ΔOAB is equilateral. **2**

(ii) Express $z_2 - z_1$ in modulus-argument form. **2**

(d) z is a complex number such that $\arg z = \frac{\pi}{3}$ and $|z| \le 2$.

(i) Show the locus of the point P representing z in the Argand diagram. **2**

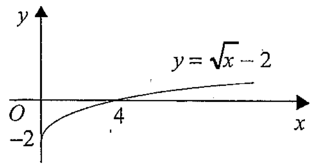
(ii) Find the possible values of the principal argument of $z - i$ for z on this locus. **2**

Student name / number ....

Marks

Question 3 **Begin a new booklet**

(a)



This diagram shows the graph of the function $f(x)=\sqrt{x}-2$.
On separate diagrams sketch the following graphs, showing clearly any intercepts on the coordinate axes and the equations of any asymptotes:

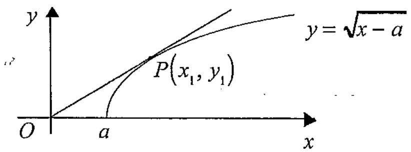
(i) $y=|f(x)|$. **1**

(ii) $y=\{f(x)\}^2$. **1**

(iii) $y=\dfrac{1}{f(x)}$. **2**

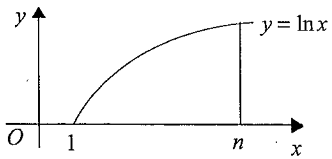
(iv) $y=\log_e f(x)$. **2**

(b)



The tangent to the curve $y=\sqrt{x-a}$, where $a>0$, at the point $P(x_1, y_1)$ on the curve passes through the origin. Find the coordinates of P. **3**

(c)



(i) Use the trapezoidal rule with n function values to approximate $\int_1^n \ln x\, dx$. **2**

(ii) Show that $\dfrac{d}{dx}(x\ln x-x)=\ln x$ and hence find the exact value of $\int_1^n \ln x\, dx$. **2**

(iii) Deduce that $\ln n! < (n+\frac{1}{2})\ln n - n + 1$. **2**

Student name / number ....

Marks

Question 4 **Begin a new booklet**

(a) The polynomial $P(x)$ leaves a remainder of 9 when divided by $(x-2)$ and a remainder of 4 when divided by $(x-3)$. Find the remainder when $P(x)$ is divided by $(x-2)(x-3)$. **2**

(b) $P\left(ct, \frac{c}{t}\right)$ and $Q(1+\cos\theta, \sin\theta)$ are points on the hyperbola $xy=c^2$, where $c>0$, and the circle $(x-1)^2+y^2=1$ respectively.

(i) Show by differentiation that the tangent to the hyperbola at P has equation $x+t^2y=2ct$ and the tangent to the circle at Q has equation $x\cos\theta+y\sin\theta=1+\cos\theta$. **3**

(ii) Deduce that PQ is tangent to both the hyperbola and the circle, with points of contact P and Q, if $t^2=\tan\theta$ and $2ct-1=\sec\theta$, where $\left(\frac{t}{c}\right)^3-4\left(\frac{t}{c}\right)+\frac{4}{c^2}=0$. **3**

(iii) By considering the graphs of $y=x^3-4x$ and $y=x^3-4x+\frac{4}{c^2}$, deduce that for every value of $c>0$ there is exactly one point on the third-quadrant branch of the hyperbola where the tangent to the hyperbola is also tangent to the circle. Show that for $c^2>\frac{3\sqrt{3}}{4}$, there are also two such points on the first-quadrant branch of the hyperbola. **3**

(iv) When $c^2=\frac{3\sqrt{3}}{4}$, the hyperbola touches the circle at $P\left(ct, \frac{c}{t}\right)$ where $\frac{t}{c}$ is a double root of the cubic equation $x^3-4x+\frac{4}{c^2}=0$. Sketch the hyperbola and the circle for $c^2=\frac{3\sqrt{3}}{4}$, showing any common tangents to the curves with their equations. Write numerical values for the coordinates of any points of contact P, Q for these tangents. **4**

Marks

Question 5 — Begin a new booklet

(a) $z = \cos\theta + i\sin\theta$

(i) Show that $z^n + z^{-n} = 2\cos n\theta$ for $n = 1, 2, 3, \ldots$. **1**

(ii) Hence show that $4\cos\theta\cos 2\theta\cos 3\theta = 1 + \cos 2\theta + \cos 4\theta + \cos 6\theta$. **2**

(iii) Hence solve $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta = 1$, giving general solutions. **3**

(b) A particle is projected vertically downwards under gravity in a medium where resistance is proportional to the speed of the particle. The terminal velocity of the particle is U ms^{-1}, and the speed of projection is equal to half this terminal velocity. At time t seconds, the particle has travelled a distance x metres, has velocity v ms^{-1} and has acceleration $\ddot{x}$ ms^{-2}.

(i) Show $\ddot{x} = \dfrac{g}{U}(U - v)$, where g ms^{-2} is the acceleration due to gravity. **2**

(ii) Show by integration that $-\dfrac{g}{U}t = \ln 2\left(1 - \dfrac{v}{U}\right)$. Hence obtain an expression for $\dfrac{v}{U}$ in terms of t. **3**

(iii) Show that $x = Ut - \dfrac{U^2}{g}\left(\dfrac{v}{U} - \dfrac{1}{2}\right)$. **2**

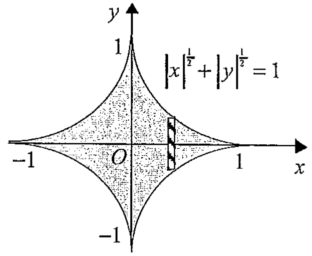
(iv) If $g = 10$ and $U = 100$, find the percentage of the terminal velocity gained during the first second of the motion, and the distance travelled during this time. **2**

Marks

Question 6 — Begin a new booklet

(a) The roots of the equation $x^3 + 3x^2 + 7x + k = 0$ are in arithmetic progression. Find the value of the constant k. **2**

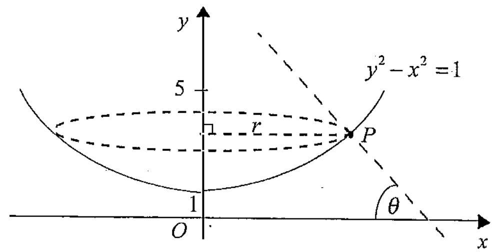
(b) The horizontal base of a solid is the area enclosed by the curve $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} = 1$.
Vertical cross sections taken perpendicular to the x-axis are squares with one side in the base.



(i) Show that the volume of the solid is given by $V = 8\int_0^1 \left(1 - \sqrt{x}\right)^4 dx$. **2**

(ii) Use the substitution $u = 1 - \sqrt{x}$ to evaluate this integral. **3**

(c)



A bowl is formed by rotating the hyperbola $y^2 - x^2 = 1$ for $1 \le y \le 5$ through 180° about the y-axis. Sometime later, a particle P of mass m moves around the inner surface of the bowl in a horizontal circle with constant angular velocity ω.

(i) Show that if the radius of the circle in which P moves is r, then the normal to the surface at P makes an angle θ with the horizontal where $\tan\theta = \dfrac{\sqrt{1 + r^2}}{r}$. **2**

(ii) Draw a diagram showing the forces on P. **1**

(iii) Find expressions for the radius r of the circle of motion and the magnitude of the reaction force between the surface and the particle in terms of m, g and ω. **3**

(iv) Find the values of ω for which the described motion of P is possible. **2**

Marks

Question 7 **Begin a new booklet**

(a) $I_n = \int_1^e (1-\ln x)^n \, dx$, $n = 0, 1, 2, \ldots$

(i) Show $I_n = -1 + nI_{n-1}$, $n = 1, 2, 3, \ldots$. **2**

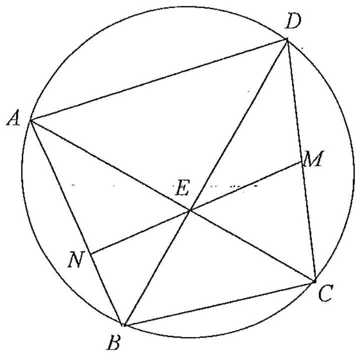
(ii) Hence evaluate $\int_1^e (1-\ln x)^3 \, dx$. **2**

(iii) Show that $\frac{I_n}{n!} = e - \sum_{r=0}^{n} \frac{1}{r!}$, $n = 1, 2, 3, \ldots$. **2**

(iv) Show that $0 \le I_n \le e-1$. **1**

(v) Deduce that $\lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{r!} = e$. **1**

(b)



ABCD is a cyclic quadrilateral. The diagonals *AC* and *BD* intersect at right angles at *E*. *M* is the midpoint of *CD*. *ME* produced meets *AB* at *N*.

(i) Copy the diagram showing the given information. Show that $ME = MC$. **3**

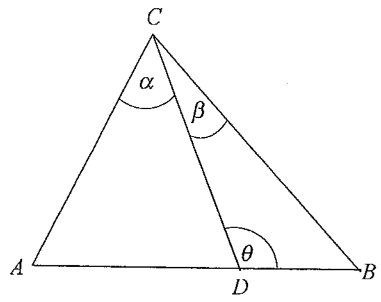
(ii) Hence show that *MN* is perpendicular to *AB*. **4**

Student name / number ....

Marks

Question 8 **Begin a new booklet**

(a)



In ΔABC, D is the point on AB that divides AB internally in the ratio $m : n$. $\angle ACD = \alpha$, $\angle BCD = \beta$ and $\angle CDB = \theta$.

(i) By using the sine rule in each of ΔCAD and ΔCDB, show that $\frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta} = \frac{m}{n}$. **4**

(ii) Hence show that $\tan\theta = \frac{(m+n)\tan\alpha\tan\beta}{m\tan\beta - n\tan\alpha}$. **3**

(b) $f(x)$ and $g(x)$ are continuous and bounded functions.

(i) By considering $\int_0^a \{\lambda f(x) + g(x)\}^2 \, dx$, $a > 0$, as a quadratic function of λ, show that **4**

$$\left\{\int_0^a f(x)g(x)\, dx\right\}^2 \le \int_0^a \{f(x)\}^2 \, dx \, . \int_0^a \{g(x)\}^2 \, dx \, .$$

(ii) Hence show that $\left\{\int_0^1 f(x)\, dx\right\}^2 \le \int_0^1 \{f(x)\}^2 \, dx$. **2**

(iii) Deduce that $\left\{\int_0^1 f(x)\, dx\right\}^4 \le \int_0^1 \{f(x)\}^4 \, dx$. **2**

Question 1

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• rearranges integrand into appropriate sum of terms	1
• finds primitive	1

Answer

$$\int \frac{(x+1)^2}{x}\,dx = \int\left(x+2+\frac{1}{x}\right)dx = \tfrac{1}{2}x^2+2x+\ln x + c$$

b. Outcomes assessed : H8, PE3, E8

Marking Guidelines

Criteria	Marks
i • writes and solves a pair of simultaneous equations for A and C	1
• writes and solves a pair of simultaneous equations for B and D	1
ii • finds and evaluates definite integral for one term	1
• finds and evaluates definite integral for second term	1

Answer

i.

$$\frac{x^3+2x^2+4x+2}{(x^2+1)(x^2+4)} \equiv \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4}$$

$$x^3+2x^2+4x+2 \equiv (Ax+B)(x^2+4)+(Cx+D)(x^2+1)$$

equating coefficients of x^3 : $1 = A+C$ (1) $\quad (2)-(1) \Rightarrow 3=3A$

equating coefficients of x : $4 = 4A+C$ (2) $\quad \therefore A=1,\ C=0$

equating coefficients of x^2 : $2 = B+D$ (3) $\quad (4)-(3) \Rightarrow 3B=0$

putting $x=0$: $2 = 4B+D$ (4) $\quad \therefore B=0,\ D=2$

ii.

$$\int_0^2 \frac{x^3+2x^2+4x+2}{(x^2+1)(x^2+4)}\,dx = \int_0^2\left(\frac{x}{(x^2+1)}+\frac{2}{(x^2+4)}\right)dx$$

$$= \left[\tfrac{1}{2}\ln(x^2+1)+\tan^{-1}\tfrac{x}{2}\right]_0^2$$

$$= \tfrac{1}{2}\ln 5+\tan^{-1}1$$

$$= \tfrac{1}{2}\ln 5+\tfrac{\pi}{4}$$

c. Outcomes assessed : H5, HE6

Marking Guidelines

Criteria	Marks
• expresses dx in terms of dt and substitutes expressions for $\cos x$ and $\sin x$ in terms of t	1
• simplifies integrand as a function of t and finds primitive in terms of t	1
• writes primitive as a function of x	1

Answer

$t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$dx = \frac{2}{1+t^2}\,dt$

$$5+4\cos x+3\sin x = \frac{5(1+t^2)+4(1-t^2)+6t}{1+t^2} = \frac{t^2+6t+9}{1+t^2} = \frac{(t+3)^2}{1+t^2}$$

$$\int\frac{1}{5+4\cos x+3\sin x}\,dx = \int\frac{1+t^2}{(t+3)^2}\cdot\frac{2}{1+t^2}\,dt = -\frac{2}{t+3}+c = -\frac{2}{3+\tan\frac{x}{2}}+c$$

d. Outcomes assessed : H8, HE6

Marking Guidelines

Criteria	Marks
• expresses du in terms of dx and converts x limits to u limits	1
• simplifies integrand as a function of u	1
• uses table of integrals to write primitive then evaluates by substitution	1

Answer

$u = \sin x \qquad x=0 \Rightarrow u=0$

$du = \cos x\,dx \qquad x=\frac{\pi}{2} \Rightarrow u=1$

$$\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^1\frac{1}{\sqrt{1+u^2}}\,du = \left[\ln\left(u+\sqrt{u^2+1}\right)\right]_0^1 = \ln(1+\sqrt{2})$$

e. Outcomes assessed : H8, HE6

Marking Guidelines

Criteria	Marks
• expresses du in terms of dx and converts x limits to u limits	1
• simplifies and rearranges integrand as function of u	1
• evaluates definite integral	1

Answer

$u = -x$

$du = -dx$

$x=-1 \Rightarrow u=1$

$x=1 \Rightarrow u=-1$

$$I = \int_{-1}^1\frac{1}{e^x+1}\,dx = \int_1^{-1}\frac{1}{e^{-u}+1}\cdot -\,du = \int_{-1}^1\frac{1}{e^{-u}+1}\,du = \int_{-1}^1\frac{e^u}{1+e^u}\,du$$

$$\therefore 2I = \int_{-1}^1\frac{1}{e^x+1}\,dx+\int_{-1}^1\frac{e^x}{1+e^x}\,dx = \int_{-1}^1\frac{1+e^x}{1+e^x}\,dx = \int_{-1}^1 1\,dx$$

$\therefore 2I = 2$

$I = 1$

Question 2

a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • finds modulus	1
ii • simplifies sum	1
iii • realises denominator to simplify quotient	1

Answer

$z_1 = 3i, \quad z_2 = 1 + i$

i. $|z_1 - z_2| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

ii. $z_1 + \overline{z}_2 = 3i + 1 - i = 1 + 2i$

iii. $\dfrac{z_1}{z_2} = \dfrac{3i(1-i)}{(1+i)(1-i)} = \frac{3}{2} + \frac{3}{2}i$

b. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • writes z in modulus-argument form	1
• writes square of z in modulus-argument form	1
• writes reciprocal of z in modulus-argument form	1
ii • uses a diagram or subtracts arguments to show points collinear	1

Answer

i. $z = 1 + i\sqrt{3}$

$z = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$z^2 = 4(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

$\dfrac{1}{z} = \frac{1}{2}\left(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})\right)$

ii. $\angle AOB = \arg z^2 - \arg\frac{1}{z} = \pi \qquad \therefore A, O, B$ are collinear

c. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • explains why sides OA, OB are equal	1
• finds size of angle at O and deduces triangle equilateral	1
ii • recognizes $\overrightarrow{AB}$ as rotation of $\overrightarrow{OB}$ and expresses $z_2 - z_1$ as multiple of z_2	1
• expresses $z_2 - z_1$ in modulus-argument form	1

Answer

$z_1 = 2(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}), \quad z_2 = 2(\cos\frac{7\pi}{15} + i\sin\frac{7\pi}{15})$

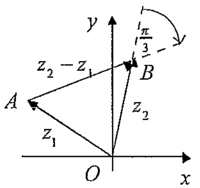
i. $|z_1| = |z_2| \quad \therefore OA = OB$

$\angle AOB = \arg z_1 - \arg z_2 = \frac{\pi}{3}$

$\therefore$ all $\angle$'s of ΔAOB are $\frac{\pi}{3}$

($\angle$ sum is π and $\angle$'s opp. equal sides are equal)

$\therefore \Delta AOB$ is equilateral.



$\overrightarrow{AB}$ represents $z_2 - z_1$.

$\overrightarrow{AB}$ is the rotation of $\overrightarrow{OB}$ clockwise by $\frac{\pi}{3}$.

$\therefore z_2 - z_1 = z_2(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}) = 2(\cos\frac{2\pi}{15} + i\sin\frac{2\pi}{15})$

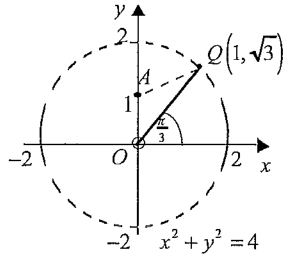
d. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • shows a ray from O making angle $\frac{\pi}{3}$ with positive x-axis, excluding O	1
• restricts ray to an interval inside the circle with centre O and radius 2.	1
ii • finds the lower bound for the required argument as a strict inequality	1
• finds the upper bound for the required argument	1

Answer

i. P represents z such that $\arg z = \frac{\pi}{3}$ and $|z| \le 2$. Locus of P is the interval OQ on the graph below, with O excluded.



ii. Point A represents the complex number i.

Using trigonometry, Q has coordinates $\left(2\cos\frac{\pi}{3}, 2\sin\frac{\pi}{3}\right)$, giving $Q\left(1, \sqrt{3}\right)$ as shown.

Gradient of AQ is $\sqrt{3} - 1$.

$\therefore -\frac{\pi}{2} < \text{Arg}(z - i) \le \tan^{-1}\left(\sqrt{3} - 1\right)$

Question 3

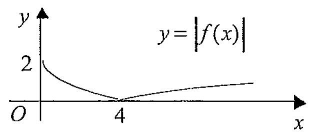
a. Outcomes assessed : E6

Marking Guidelines

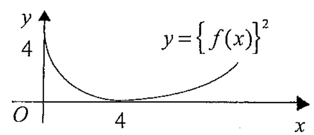
Criteria	Marks
i • sketches curve with correct shape and intercepts	1
ii • sketches curve with correct shape and intercepts	1
iii • shows correct shape and y-intercept for branch to left of vertical asymptote at $x = 4$	1
• shows correct shape and position of branch to right of vertical asymptote	1
iv • shows correct shape, position and behaviour near vertical asymptote at $x = 4$	1
• shows x-intercept	1

Answer

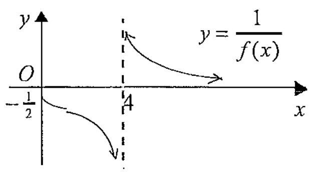
i.



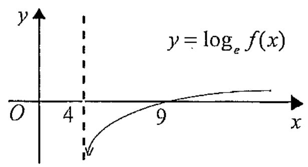
ii.



iii.



iv.



b. Outcomes assessed : E6

Marking Guidelines

Criteria	Marks
• uses differentiation to find the gradient of the tangent at P	1
• uses coordinates of O and P to find gradient of OP and hence writes equation for x_1	1
• solves this equation to find coordinates of P	1

Answer

$$y=\sqrt{x-a}$$
$$\frac{dy}{dx}=\frac{1}{2\sqrt{x-a}}$$
$\therefore$ tangent at P has gradient $\dfrac{1}{2\sqrt{x_1-a}}$

$$\therefore \frac{1}{2\sqrt{x_1-a}}=\frac{y_1}{x_1}$$
$$x_1=2y_1\sqrt{x_1-a}$$
$$x_1=2(x_1-a)$$
$$x_1=2x_1-2a$$
Hence P has coordinates $\left(2a\,,\sqrt{a}\right)$

c. Outcomes assessed : H8, PE3

Marking Guidelines

Criteria	Marks
i • writes expression for approximate area using trapezoidal rule	1
• simplifies this expression	1
ii • differentiates given expression	1
• uses fact that integration is the inverse operation to evaluate required definite integral	1
iii • compares total area enclosed by trapezia with area under curve	1
• uses this to write and simplify inequality	1

Answer

i.
$$\int_1^n \ln x\,dx$$
$$\approx \tfrac{1}{2}\left\{\ln 1+2\left[\ln 2+\ln 3+\ldots+\ln(n-1)\right]+\ln n\right\}$$
$$=\ln 1+\ln 2+\ln 3+\ldots+\ln n-\tfrac{1}{2}(\ln 1+\ln n)$$
$$=\ln n!\,-\tfrac{1}{2}\ln n$$

ii.
$$\frac{d}{dx}\left(x\ln x-x\right)=1\,.\ln x+x\,.\frac{1}{x}-1$$
$$=\ln x$$
$$\left[x\ln x-x\right]_1^n=\int_1^n \ln x\,dx$$
$$\therefore \int_1^n \ln x\,dx=n\ln n-n+1$$

iii. The total area of the trapezia fitted under the curve is less than the area under the curve.
$$\therefore \ln n!-\tfrac{1}{2}\ln n<n\ln n-n+1$$
$$\ln n!<(n+\tfrac{1}{2})\ln n-n+1$$

Question 4

a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
• writes division transformation of $P(x)$ when divided by $(x-2)(x-3)$ with remainder $(ax+b)$	1
• uses remainder theorem to write simultaneous equations for a, b then finds required remainder	1

Answer

Using the division transformation, $P(x)\equiv(x-2)(x-3)Q(x)+ax+b$ for some polynomial $Q(x)$ and real constants a, b, where $(ax+b)$ is the remainder when $P(x)$ is divided by $(x-2)(x-3)$.

Then $\left.\begin{array}{l}P(2)=9\Rightarrow 2a+b=9\\P(3)=4\Rightarrow 3a+b=4\end{array}\right\}$ $\therefore \begin{array}{l}a=-5\\b=19\end{array}$ $\therefore$ remainder is $(-5x+19)$.

b. Outcomes assessed : E3, E4

Marking Guidelines

Criteria	Marks
i • finds equation of tangent to hyperbola by differentiation	1
• finds gradient of tangent to circle in terms of θ	1
• completes equation of tangent to circle using appropriate trig. identity	1
ii • compares coefficients for two forms of equation of PQ to obtain results for $\tan\theta$, $\sec\theta$	1
• obtains quartic equation for t	1
• factors then rearranges to get required cubic equation	1
iii • graphs $y=x^3-4x$ showing intercepts and stationary point in 4th quadrant	1
• compares vertical translations to deduce existence of such a point on 3rd quad. branch	1
• uses turning point to deduce existence of such points on 1st quad. branch for stated c	1
iv • sketches hyperbola and circle, touching in first quadrant, with two common tangents	1
• gives coordinates of point where curves touch and equation of this common tangent	1
• gives coordinates of point of contact on hyperbola for second common tangent	1
• gives equation of this tangent and coordinates of its point of contact with circle	1

Answer

i.

$x=ct$ $\quad y=\frac{c}{t}$

$\dfrac{dx}{dt}=c$ $\quad \dfrac{dy}{dt}=-\dfrac{c}{t^2}$

$\therefore \dfrac{dy}{dx}=-\dfrac{c}{t^2}\div c=-\dfrac{1}{t^2}$

Tangent to hyperbola has equation
$$y-\tfrac{c}{t}=-\tfrac{1}{t^2}(x-ct)$$
$$x+t^2y=2ct$$

$x=1+\cos\theta$ $\quad y=\sin\theta$

$\dfrac{dx}{d\theta}=-\sin\theta$ $\quad \dfrac{dy}{d\theta}=\cos\theta$

$\therefore \dfrac{dy}{dx}=-\dfrac{\cos\theta}{\sin\theta}$

Tangent to circle has equation
$$y-\sin\theta=-\frac{\cos\theta}{\sin\theta}\left(x-1-\cos\theta\right)$$
$$x\cos\theta+y\sin\theta=\cos^2\theta+\sin^2\theta+\cos\theta$$
$$x\cos\theta+y\sin\theta=1+\cos\theta$$

ii. When these two tangents are in fact the same line, PQ is tangent to both curves. Comparing the equations $x+t^2y=2ct$ and $x+y\tan\theta=\sec\theta+1$ (rearrangement of tangent to circle at Q), these equations give the same line when $t^2=\tan\theta$ and $2ct-1=\sec\theta$.

Then
$$1+\tan^2\theta=\sec^2\theta\ \Rightarrow 1+t^4=(2ct-1)^2$$
$$t^4-4c^2t^2+4ct=0$$
$$t(t^3-4c^2t+4c)=0$$

$$t\neq 0\ \Rightarrow t^3-4c^2t+4c=0$$
$$\therefore\quad \left(\tfrac{t}{c}\right)^3-4\left(\tfrac{t}{c}\right)+\tfrac{4}{c^2}=0 \quad *$$

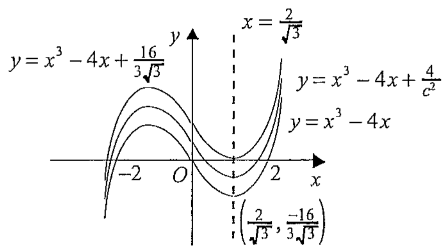
iii.

$y = x^3 - 4x$

$\dfrac{dy}{dx} = 3x^2 - 4$

$\dfrac{dy}{dx} = 0 \Rightarrow x = \pm\frac{2}{\sqrt{3}}$

$x = \frac{2}{\sqrt{3}} \Rightarrow y = -\frac{16}{3\sqrt{3}}$

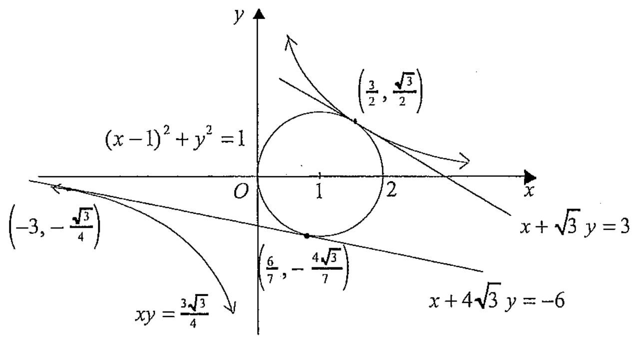


The graph of $y = x^3 - 4x + k$, $k > 0$, is an upward vertical translation of $y = x^3 - 4x$.
Hence for all $c > 0$, $y = x^3 - 4x + \frac{4}{c^2}$ has exactly one negative x-intercept corresponding to one negative t value satisfying *, thus giving exactly one point P on the third-quadrant branch of the hyperbola where the tangent to the hyperbola is also tangent to the circle.
For $\frac{4}{c^2} < \frac{16}{3\sqrt{3}}$, $y = x^3 - 4x + \frac{4}{c^2}$ also has two distinct positive x-intercepts. Hence for $c^2 > \frac{3\sqrt{3}}{4}$, there are two distinct positive t values satisfying *, giving two such points P on the first-quadrant branch of the hyperbola.

iv. When $c^2 = \frac{3\sqrt{3}}{4}$, $x^3 - 4x + \frac{4}{c^2} = 0$ becomes $x^3 - 4x + \frac{16}{3\sqrt{3}} = 0$. From the graph, this equation has a double root $\frac{2}{\sqrt{3}}$, and a third root $\frac{-4}{\sqrt{3}}$ (since the sum of the roots is zero).

$\frac{t}{c} = \frac{2}{\sqrt{3}} \Rightarrow ct = \frac{3}{2}$, $\frac{c}{t} = \frac{\sqrt{3}}{2}$ $\therefore$ curves touch at $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$ with common tangent $x + \sqrt{3}\, y = 3$.

$\frac{t}{c} = \frac{-4}{\sqrt{3}} \Rightarrow ct = -3$, $\frac{c}{t} = -\frac{\sqrt{3}}{4}$ and $\sec\theta = 2ct - 1 = -7$, $\tan\theta = t^2 = 4\sqrt{3}$ $\therefore$ points of contact of second common tangent $x + 4\sqrt{3}\, y = -6$ are $\left(-3, -\frac{\sqrt{3}}{4}\right)$ on the hyperbola and $\left(\frac{6}{7}, -\frac{4\sqrt{3}}{7}\right)$ on the circle.



Question 5

a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • uses de Moivre's theorem to show result	1
ii • expands $(z^1 + z^{-1})(z^2 + z^{-2})(z^3 + z^{-3})$	1
• rearranges, regroups and applies result from i. to obtain required trig. identity	1
iii • uses double angle formula and identity from ii. to simplify equation	1
• obtains general solution for $\cos\theta = 0$, or obtains all solutions for $-\pi < \theta \le \pi$	1
• obtains remaining general solutions	1

Answer

i. $z = \cos\theta + i\sin\theta$. By de Moivre's theorem, $z^n = \cos n\theta + i\sin n\theta$ for $n = 1, 2, 3, \ldots$.
Then $z^{-n} = \cos(-n\theta) + i\sin(-n\theta) \Rightarrow z^{-n} = \cos n\theta - i\sin n\theta$
Hence $z^n + z^{-n} = 2\cos n\theta$

ii. $(z^1 + z^{-1})(z^2 + z^{-2})(z^3 + z^{-3}) = (z^3 + z^{-3} + z^1 + z^{-1})(z^3 + z^{-3})$
$= 2 + z^2 + z^{-2} + z^4 + z^{-4} + z^6 + z^{-6}$
$\therefore 2\cos\theta \,.\, 2\cos 2\theta \,.\, 2\cos 3\theta = 2 + 2\cos 2\theta + 2\cos 4\theta + 2\cos 6\theta$
$4\cos\theta\cos 2\theta\cos 3\theta = 1 + \cos 2\theta + \cos 4\theta + \cos 6\theta$

iii.
$\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta = 1$
$2\cos^2\theta + 2\cos^2 2\theta + 2\cos^2 3\theta - 2 = 0$
$(1 + \cos 2\theta) + (1 + \cos 4\theta) + (1 + \cos 6\theta) - 2 = 0$
$1 + \cos 2\theta + \cos 4\theta + \cos 6\theta = 0$
$4\cos\theta\cos 2\theta\cos 3\theta = 0$

$\cos\theta = 0 \Rightarrow \theta = 2m\pi \pm \frac{\pi}{2} = (4m \pm 1)\frac{\pi}{2}$
$\therefore \theta = (4m \pm 1)\frac{\pi}{2},\ (4m \pm 1)\frac{\pi}{4},\ (4m \pm 1)\frac{\pi}{6}$
for $m = 0, \pm 1, \pm 2, \ldots$

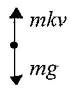
b. Outcomes assessed : E5

Marking Guidelines

Criteria	Marks
i • uses Newton's 2nd law to derive equation of motion	1
• uses $\ddot{x} \to 0$ as $v \to U$ to express equation of motion in required form	1
ii • finds primitive function for t	1
• evaluates constant of integration to establish required expression for t in terms of v	1
• rearranges to find expression for v in terms of t	1
iii • integrates to find x in terms of t	1
• substitutes and rearranges to obtain required expression for x	1
iv • calculates percentage of terminal velocity gained in first second	1
• calculates distance travelled in first second.	1

Answer

i.



i. By Newton's 2nd Law,
$m\ddot{x} = mg - mkv$
$\ddot{x} = g - kv$

$\ddot{x} \to 0$ as $v \to \frac{g}{k} \Rightarrow U = \frac{g}{k}$ $\therefore k = \frac{g}{U}$

$\therefore \ddot{x} = \frac{g}{U}(U - v)$

ii.

$$\frac{dv}{dt}=\frac{g}{U}(U-v)$$

$$-\frac{g}{U}\frac{dt}{dv}=-\frac{1}{U-v}$$

$$-\frac{g}{U}t=\ln A(U-v),\quad A\text{ const.}$$

$$\left.\begin{aligned}t&=0\\ v&=\tfrac{1}{2}U\end{aligned}\right]\Rightarrow A=\frac{2}{U}$$

$$\therefore\ -\frac{g}{U}t=\ln 2\left(1-\frac{v}{U}\right)$$

$$e^{-\frac{g}{U}t}=2\left(1-\frac{v}{U}\right)$$

$$\therefore \frac{v}{U}=1-\tfrac{1}{2}e^{-\frac{g}{U}t}$$

iii.

$$v=U-\tfrac{1}{2}U\,e^{-\frac{g}{U}t}$$

$$x=Ut+\frac{U^2}{2g}e^{-\frac{g}{U}t}+c,\quad c\text{ const}$$

$$\left.\begin{aligned}t&=0\\ x&=0\end{aligned}\right]\Rightarrow 0=0+\frac{U^2}{2g}+c\qquad \therefore c=-\frac{U^2}{2g}$$

$$x=Ut+\frac{U^2}{g}\left(\tfrac{1}{2}e^{-\frac{g}{U}t}-\tfrac{1}{2}\right)$$

$$\therefore\ x=Ut-\frac{U^2}{g}\left(\frac{v}{U}-\frac{1}{2}\right)$$

iv. $t=1\ \Rightarrow\ \frac{v}{U}-\frac{1}{2}=\frac{1}{2}\left(1-e^{-0\cdot 1}\right)\approx 0\cdot 04758$

$$x=100-1000\times\left(\frac{v}{U}-\frac{1}{2}\right)\approx 52\cdot 4187$$

$\therefore$ particle has gained $4\cdot 8\%$ of its terminal velocity and travelled $52\cdot 4$ metres during the first second.

Question 6

a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
• uses relationship between coefficients and sum of roots to find one root	1
• substitutes this root into the equation to find the value of k	1

Answer

Let $x^3+3x^2+7x+k=0$ have roots $\alpha-d,\ \alpha,\ \alpha+d$. Then considering the sum of the roots

$3\alpha=-3\qquad \therefore\ \alpha=-1$

Then $(-1)^3+3(-1)^2+7(-1)+k=0\ \Rightarrow k=5$

b. Outcomes assessed : E7

Marking Guidelines

Criteria	Marks
i • finds area of square cross section in terms of x	1
• expresses volume as limiting sum of slices and hence as integral	1
ii • expresses dx in terms of du	1
• converts x limits to u limits and simplifies new integrand after substitution	1
• evaluates resulting definite integral	1

Answer

i. Area of square cross section is $A=(2y)^2=4\left(1-|x|^{\frac{1}{2}}\right)^4$, since $y=\left(1-|x|^{\frac{1}{2}}\right)^2$, $-1\le x\le 1$.

Hence $V=\lim\limits_{\delta x\to 0}\sum\limits_{x=-1}^{1}4\left(1-|x|^{\frac{1}{2}}\right)^4\delta x=8\lim\limits_{\delta x\to 0}\sum\limits_{x=0}^{1}\left(1-x^{\frac{1}{2}}\right)^4\delta x=8\int_0^1\left(1-\sqrt{x}\right)^4dx$ (using symmetry)

ii.

$$u=1-\sqrt{x}$$
$$x=(1-u)^2$$
$$dx=-2(1-u)du$$

$$x=0\Rightarrow u=1$$
$$x=1\Rightarrow u=0$$

$$\therefore V=8\int_1^0 u^4.-2(1-u)\,du$$
$$=16\int_0^1(u^4-u^5)\,du$$
$$=16\left[\tfrac{1}{5}u^5-\tfrac{1}{6}u^6\right]_0^1$$
$$=\tfrac{8}{15}$$

c. Outcomes assessed : E5

Marking Guidelines

Criteria	Marks
i • finds gradient of tangent by differentiation	1
• finds gradient of normal at P and hence deduces required expression for $\tan\theta$	1
ii • draws diagram showing forces on P	1
iii • resolves vertically and horizontally to find simultaneous equations	1
• finds r in terms of g and ω	1
• finds N in terms of m, g and ω	1
iv • considers expressions for r, N and the height of the bowl to find limits for ω	1

Answer

i. $y^2-x^2=1,\quad P\left(r,\ \sqrt{1+r^2}\right)$.

$$2y\frac{dy}{dx}-2x=0$$

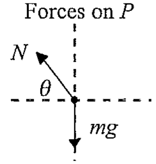
$$\frac{dy}{dx}=\frac{x}{y}$$

$\therefore$ normal at P has gradient

$$-\frac{\sqrt{1+r^2}}{r}$$

$$\therefore \tan\theta=\frac{\sqrt{1+r^2}}{r}$$

ii. Forces on P



iii. Resolving vertically and horizontally, by Newton's 2nd law

$N\sin\theta=mg\qquad(1)$

$N\cos\theta=mr\omega^2\qquad(2)$

$$\frac{(1)}{(2)}\Rightarrow \tan\theta=\frac{g}{r\omega^2}$$

$$\frac{\sqrt{1+r^2}}{r}=\frac{g}{r\omega^2}$$

$$\therefore r=\sqrt{\frac{g^2}{\omega^4}-1}$$

$$r=\frac{\sqrt{g^2-\omega^4}}{\omega^2}$$

Then from (2)

$$N^2=m^2r^2\omega^4\sec^2\theta$$
$$=m^2r^2\omega^4(1+\tan^2\theta)$$
$$=m^2\omega^4(r^2+r^2\tan^2\theta)$$
$$=m^2\omega^4\left(\frac{g^2}{\omega^4}-1+\frac{g^2}{\omega^4}\right)$$
$$=m^2(2g^2-\omega^4)$$
$$\therefore N=m\sqrt{2g^2-\omega^4}$$

iv. Considering expressions for r and N, $\omega^4\le g^2\quad \therefore \omega\le\sqrt{g}$

Also $y\le 5\ \Rightarrow \sqrt{1+r^2}\le 5\ \Rightarrow \frac{g}{\omega^2}\le 5\qquad \therefore \sqrt{\frac{g}{5}}\le\omega\le\sqrt{g}$

Question 7

a. Outcomes assessed : E8, E9

Marking Guidelines

Criteria	Marks
i • executes integration by parts	1
• evaluates numerical term and simplifies new definite integral to obtain reduction formula	1
ii • finds value of I_0	1
• uses reduction formula to evaluate required integral	1
iii • obtains reduction formula for $\frac{1}{r!}I_r$, $r=1,2,\ldots,n$	1
• obtains required result by summation and simplification	1
iv • shows integrand lies between 0 and 1 for $1\le x\le e$ and deduces required inequality	1
v • shows 0 is limiting value of $\frac{1}{n!}I_n$ as $n\to\infty$ then deduces required result	1

Answer

i. $I_n=\int_1^e (1-\ln x)^n\,dx$, $\quad n=0,1,2,\ldots$

For $n\ge 1$,

$$I_n=\Big[x(1-\ln x)^n\Big]_1^e-\int_1^e x\,.\,n(1-\ln x)^{n-1}(-\tfrac{1}{x})dx$$

$$=-1+n\int_1^e(1-\ln x)^{n-1}\,dx$$

$\therefore I_n=-1+nI_{n-1}$, $\quad n=1,2,3,\ldots$

ii. $I_0=\int_1^e 1\,dx=e-1$

$$\begin{aligned}I_3&=-1+3I_2\\&=-1+3(-1+2I_1)\\&=-4+6(-1+1I_0)\\&=-10+6(e-1)\\&=-16+6e\end{aligned}$$

iii. $\dfrac{I_r}{r!}=\dfrac{-1}{r!}+\dfrac{rI_{r-1}}{r!}$, $r=1,2,\ldots,n$

$$\frac{I_r}{r!}=\frac{-1}{r!}+\frac{I_{r-1}}{(r-1)!}$$

$$\sum_{r=1}^{n}\frac{I_r}{r!}=-\sum_{r=1}^{n}\frac{1}{r!}+\sum_{r=1}^{n}\frac{I_{r-1}}{(r-1)!}$$

$$\sum_{r=1}^{n}\frac{I_r}{r!}=-\sum_{r=1}^{n}\frac{1}{r!}+\sum_{r=0}^{n-1}\frac{I_r}{r!}$$

$$\therefore \frac{I_n}{n!}=-\sum_{r=1}^{n}\frac{1}{r!}+\frac{I_0}{0!}$$

$$=-\sum_{r=1}^{n}\frac{1}{r!}+\frac{-1}{0!}+\frac{e}{0!}$$

$$=e-\sum_{r=0}^{n}\frac{1}{r!}$$

iv. $1\le x\le e\ \Rightarrow\ 0\le \ln x\le 1 \Rightarrow 0\le(1-\ln x)^n\le 1$

$$\therefore 0\le\int_1^e(1-\ln x)^n\,dx\le\int_1^e 1\,dx$$

$$0\le I_n\le e-1$$

v. $\quad 0\le \dfrac{I_n}{n!}\le\dfrac{e-1}{n!}$

But $\lim\limits_{n\to\infty}\dfrac{e-1}{n!}=0 \qquad \therefore \lim\limits_{n\to\infty}\dfrac{I_n}{n!}=0$

Then $\lim\limits_{n\to\infty}\left(e-\sum\limits_{r=0}^{n}\dfrac{1}{r!}\right)=\lim\limits_{n\to\infty}\dfrac{I_n}{n!}=0$

$$\therefore\quad e-\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}=0$$

$$\therefore\quad \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}=e$$

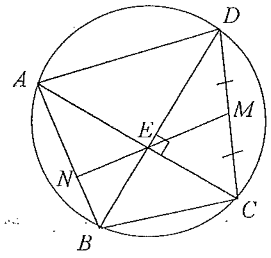
b. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • copies diagram, shows given information, realises a circle can be drawn through C, D, E	1
• explains why CD is a diameter of circle CDE.	1
• deduces that M is the centre of circle CDE and hence ME, MC are radii.	1
ii • uses i. to deduce $\angle ECM=\angle CEM$	1
• uses equality of vertically opposite angles to deduce $\angle NEA=\angle ECD$	1
• uses equality of angles subtended by same arc at circumference to deduce $\angle NAE=\angle EDC$	1
• uses angle sum property of a triangle to complete proof	1

Answer

i.



A unique circle can be drawn through any set of 3 non-collinear points.
Consider the circle that passes through C, D and E.
Since $\angle CED=90°$, CD is a diameter of this circle.
Also, given M is the midpoint of DC, M is the centre of circle CDE.
$\therefore$ $ME=MC$ (*radii of a circle are equal*)

ii.

$\angle ECM=\angle CEM$	(*$\angle$'s opp. equal sides are equal in ΔMEC*)
$\angle NEA=\angle CEM$	(*vert. opp. $\angle$'s are equal*)
$\therefore \angle NEA=\angle ECM=\angle ECD$	(*D, M, C collinear*)
Also $\angle BAC=\angle BDC$	(*$\angle$'s subtended at the circumference by the same arc BC are equal*)
$\therefore \angle NAE=\angle EDC$	(*$\angle NAE$, $\angle BAC$ same angle; $\angle EDC$, $\angle BDC$ same angle*)
$\therefore \angle ANE=\angle DEC=90°$	(*third $\angle$'s of Δ's NAE, EDC also equal since $\angle$ sum of each is 180°*)
$\therefore MN\perp AB$	

Question 8

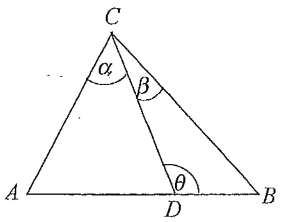
a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i • uses sine rule in ΔCAD to relate sides CD, AD and opposite angles	1
• uses sine rule in ΔCDB to relate sides CD, DB and opposite angles	1
• combines resulting equalities to obtain ratio $AD : DB$ in terms of sine ratios of angles	1
• uses internal division information and trig. identity for sine to deduce required result	1
ii • expands sine of a sum and of a difference	1
• divides by product of cosine ratios to rearrange result in terms of tangent ratios	1
• rearranges to get required result	1

Answer

i.



In ΔCAD, by ext. $\angle$ theorem,
$\angle CAD = \theta - \alpha$

$$\therefore \frac{CD}{\sin(\theta-\alpha)} = \frac{AD}{\sin\alpha} \quad (1)$$

In ΔCDB, by $\angle$ sum 180°,
$\angle CBD = 180° - (\theta+\beta)$

$$\therefore \frac{CD}{\sin\{180° - (\theta+\beta)\}} = \frac{DB}{\sin\beta} \quad (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{\sin(\theta+\beta)}{\sin(\theta-\alpha)} = \frac{AD\sin\beta}{DB\sin\alpha}$$

$$\therefore \frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta} = \frac{AD}{DB} = \frac{m}{n}$$

ii. $n\sin\alpha(\sin\theta\cos\beta + \cos\theta\sin\beta) = m\sin\beta(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

Dividing both sides by $\cos\alpha\cos\beta\cos\theta$ gives

$n\tan\alpha(\tan\theta + \tan\beta) = m\tan\beta(\tan\theta - \tan\alpha)$

$(n+m)\tan\alpha\tan\beta = \tan\theta(m\tan\beta - n\tan\alpha)$

$$\tan\theta = \frac{(n+m)\tan\alpha\tan\beta}{(m\tan\beta - n\tan\alpha)}$$

b. Outcomes assessed : H5, PE3, E9

Marking Guidelines

Criteria	Marks
i • realises that the definite integral cannot be negative for any real λ	1
• expands the integrand to write a quadratic expression in λ	1
• finds the discriminant of this quadratic expression	1
• uses $\Delta \le 0$ to deduce the required result	1
ii • substitutes $a=1$ and $g(x)=1$	1
• evaluates integral involving $\{g(x)\}^2$ to obtain required result	1
iii • squares both sides of result from ii.	1
• applies result from ii. a second time with $f(x)$ replaced by $\{f(x)\}^2$	1

Answer

i. Since $a>0$, $\int_0^a \{\lambda f(x) + g(x)\}^2\,dx \ge 0$ for all real λ.

$$\int_0^a \{\lambda f(x)+g(x)\}^2\,dx = \lambda^2\int_0^a \{f(x)\}^2\,dx + 2\lambda\int_0^a f(x)g(x)\,dx + \int_0^a \{g(x)\}^2\,dx$$

Considered as a quadratic in λ, this expression has discriminant $\Delta \le 0$.

$$\therefore 4\left\{\int_0^a f(x)g(x)\,dx\right\}^2 - 4\int_0^a \{f(x)\}^2\,dx \,.\, \int_0^a \{g(x)\}^2\,dx \le 0$$

$$\therefore \left\{\int_0^a f(x)g(x)\,dx\right\}^2 \le \int_0^a \{f(x)\}^2\,dx \,.\, \int_0^a \{g(x)\}^2\,dx$$

ii. Let $a=1$ and $g(x)=1$. Then $\int_0^1 1^2\,dx = 1 \Rightarrow \left\{\int_0^1 f(x)\,dx\right\}^2 \le \int_0^1 \{f(x)\}^2\,dx$

iii. $\left\{\int_0^1 f(x)\,dx\right\}^4 \le \left\{\int_0^1 \{f(x)\}^2\,dx\right\}^2 \le \int_0^1 \left\{\{f(x)\}^2\right\}^2\,dx \qquad \therefore \left\{\int_0^1 f(x)\,dx\right\}^4 \le \int_0^1 \{f(x)\}^4\,dx$