NSW INDEPENDENT SCHOOLS

2016
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A reference sheet is provided
- All necessary working should be shown in Question 11 – 16
- Write your student number and/or name at the top of every page

Total marks – 100

Section I - Pages 2 – 5

10 marks

Attempt Questions 1 - 10

Allow about 15 minutes for this section

Section II - Pages 6 – 11

90 marks

Attempt Questions 11 – 16

Allow about 2 hours 45 minutes for this section

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

Student name / number ____________

Marks

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

1 The function $f(x)$ is given by $f(x)=\frac{1-x}{1-\sqrt{x}}$ for $x\neq1$, and $f(1)=k$ for some constant k. If $f(x)$ is continuous at $x=1$, what is the value of k? 1

(A) 0

(B) 1

(C) 2

(D) 4

2 If $\sec x+\tan x=\frac{1}{2}$, what is the value of $\tan x$? 1

(A) $-\frac{4}{5}$

(B) $-\frac{3}{4}$

(C) $-\frac{2}{3}$

(D) $-\frac{1}{2}$

3 Which of the following is an expression for $\frac{d}{dx}\tan^{-1}\left(\frac{x+1}{x-1}\right)$? 1

(A) $\frac{x^2-2x+1}{x^2+2x+2}$

(B) $\frac{x^2-2x+1}{2(1+x^2)}$

(C) $\frac{1}{1+x^2}$

(D) $\frac{-1}{1+x^2}$

Marks

4 What is the total number of different arrangements that can be made using 3 of the 6 letters of the word PANAMA ? 1

(A) 20

(B) 34

(C) 68

(D) 120

5 If $1+i$ is one root of the equation $z^2-z+(1-i)=0$, what is the value of the other root ? 1

(A) $1-i$

(B) $-i$

(C) i

(D) $1+i$

6 What is the multiplicity of the root 1 of the equation $20x^6-30x^5+30x-20=0$? 1

(A) 1

(B) 2

(C) 3

(D) 4

7 If $x^2y-xy^2=6$, which of the following is an expression for $\frac{dy}{dx}$? 1

(A) $\dfrac{y^2-2xy}{x^2-2xy}$

(B) $\dfrac{2xy-y^2}{x^2-2xy}$

(C) $\dfrac{x^2-2xy}{2xy-y^2}$

(D) $\dfrac{x^2-2xy}{y^2-2xy}$

Marks

8 What is the value of the eccentricity e of the ellipse $\frac{x^2}{k^2}+\frac{y^2}{k^2-1}=1$, where $k>1$? 1

(A) $\dfrac{1}{k}$

(B) $\dfrac{1}{k-1}$

(C) $\dfrac{k-1}{k}$

(D) $\dfrac{k}{k+1}$

9 Which of the following is an expression for $\int \frac{1}{1+\cos x+\sin x}\,dx$ after the substitution $t=\tan\frac{x}{2}$? 1

(A) $\displaystyle\int \frac{1}{2+2t-t^2}\,dt$

(B) $\displaystyle\int \frac{2}{2+2t-t^2}\,dt$

(C) $\displaystyle\int \frac{1}{2(1+t)}\,dt$

(D) $\displaystyle\int \frac{1}{1+t}\,dt$

10 The horizontal base of a solid is the circle $x^2+y^2=1$. Each cross section taken perpendicular to the x axis is a triangle with one side in the base of the solid. The length of this triangle side is equal to the altitude of the triangle through the opposite vertex. Which of the following is an expression for the volume of the solid ? 1

(A) $\frac{1}{2}\int_{-1}^{1}(1-x^2)\,dx$

(B) $\int_{-1}^{1}(1-x^2)\,dx$

(C) $\frac{3}{2}\int_{-1}^{1}(1-x^2)\,dx$

(D) $2\int_{-1}^{1}(1-x^2)\,dx$

Student name / number ____________

Marks

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer the questions in writing booklets provided. Use a separate writing booklet for each question. In Questions 11-16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet.**

(a) If $z=1+2i$, find in the form $a+ib$, where a and b are real, the values of

(i) $iz+\overline{z}$ 1

(ii) $\frac{1}{z}$ 1

(b) The equation $x^3+2x+1=0$ has roots α, β and γ. Find in expanded form the monic cubic equation with roots $\alpha+1$, $\beta+1$ and $\gamma+1$. 2

(c) Find $\int \frac{\cos 2x}{\cos^2 x}\,dx$. 2

(d) The equation $z^5-1=0$ has roots $1, \omega, \omega^2, \omega^3, \omega^4$ where $\omega=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$.

(i) Show that $1+\omega+\omega^2+\omega^3+\omega^4=0$. 1

(ii) Show that $\left(\omega+\frac{1}{\omega}\right)^2+\left(\omega+\frac{1}{\omega}\right)-1=0$. 1

(iii) Hence show that $\cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}$. 2

(e) Consider the curve $y=\frac{x^2(x-3)}{(x-2)^2}$.

(i) Write the equation of the curve in the form $y=ax+b+\frac{c}{(x-2)^2}$. 2

(ii) Find the coordinates and nature of any turning points, then sketch the curve showing clearly any intercepts on the coordinate axes, turning points and asymptotes. 3

Student name / number ____________

Marks

Question 12 (15 marks) **Use a separate writing booklet.**

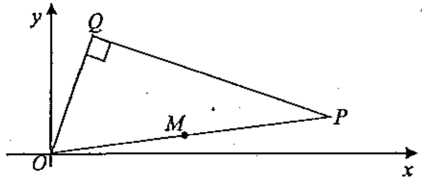
(a) The point $P(x,y)$ representing the complex number z moves in the Argand diagram so that $|z-6|=|z+2i|$.

(i) Show that the locus of P has equation $3x+y-8=0$. 2

(ii) Find the minimum value of $|z|$ as P moves on this locus. 1

(b) Use the substitution $u=e^x+1$ to find $\int \frac{e^{2x}}{(e^x+1)^2}\,dx$. 3

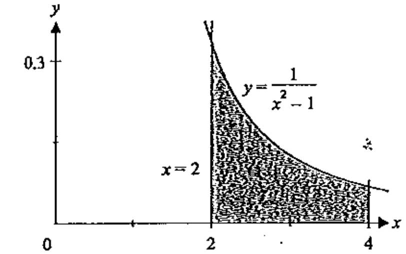
(c) In the Argand diagram below, OPQ is a triangle which is right-angled at Q and in which $QP=k.OQ$ for some constant $k>0$. M is the midpoint of OP.



(i) If $\overrightarrow{OP}=z$ and $\overrightarrow{OQ}=w$, show that $\overrightarrow{OM}=\frac{1}{2}(1-ki)w$. 2

(ii) Express $\overrightarrow{MQ}$ in terms of w and hence show that $OM=MQ$. 2

(d)



The region bounded by the curve $y=\frac{1}{x^2-1}$ and the x axis between $x=2$ and $x=4$ is rotated through one revolution about the line $x=2$.

(i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by $V=2\pi\int_2^4 \frac{x-2}{x^2-1}\,dx$. 2

(ii) Hence find the value of V in simplest exact form. 3

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Marks

Question 13 (15 marks) **Use a separate writing booklet.**

(a) Use the substitution $u=\frac{1}{x}$ to evaluate $\int_{\frac{1}{e}}^{e}\frac{\log_e x}{(1+x)^2}\,dx$. 3

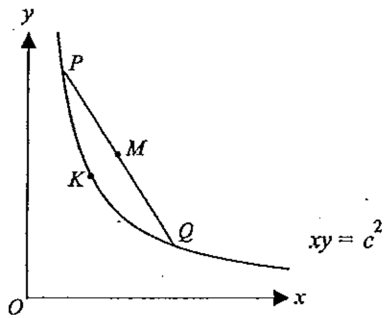
(b) If $a>0,\ b>0,\ c>0$ are real numbers, show that

(i) $(a+b+c)^2\geq 3(ab+bc+ca)$ 2

(ii) $(ab+bc+ca)^2\geq 3abc(a+b+c)$ 1

(iii) $(a+b+c)^3\geq 27abc$ 1

(c)



$K\left(ck,\frac{c}{k}\right)$, $k>0$, is a fixed point on the rectangular hyperbola $xy=c^2$.
$P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right)$, where $p>0$ and $q>0$, are two variable points which move on the rectangular hyperbola so that the parameter values p, k, q are in geometric progression.

(i) Show that the chord PQ is parallel to the tangent to the hyperbola at K. 2

(ii) If M is the midpoint of chord PQ, find in terms of k the equation of the locus of M as P and Q move on the hyperbola. 2

(d) For $n=1,2,3,\ldots$ the sequence of numbers D_n is defined by the following : 4
$D_1=0,\ D_2=1$ and, for $n\geq 3$, $D_n=(n-1)(D_{n-1}+D_{n-2})$.

Use Mathematical Induction to show that $D_n=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$ for $n\geq 1$.

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Marks

Question 14 (15 marks) **Use a separate writing booklet.**

(a) The equation $x^3+px+q=0$ has roots α, β and γ. Show that $\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma$. 3

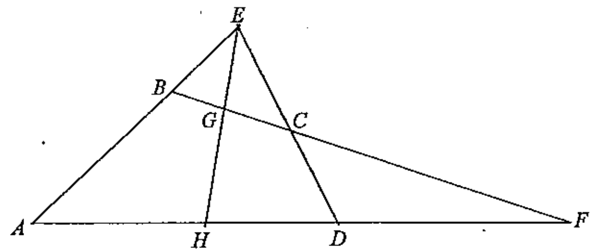
(b) 10 fair dice are rolled together. Find as a fraction in its simplest form the probability

(i) there are equal numbers of even scores and odd scores. 1

(ii) there are more even scores than odd scores. 1

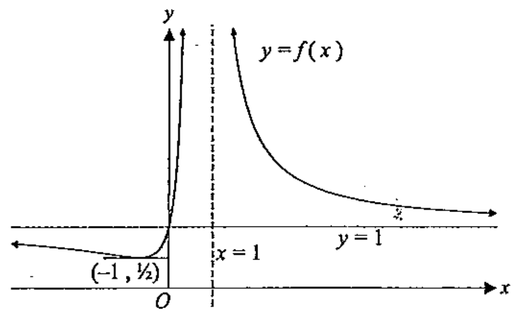
(iii) the product of all 10 scores is even. 2

(c)



$ABCD$ is a cyclic quadrilateral. AB produced and DC produced meet at E. AD produced and BC produced meet at F. EGH bisects $\angle AED$ where H lies on AD and G lies on BC. Copy the diagram and show that $FG=FH$. 4

(d)



The diagram shows the graph of the function $y=f(x)$, where $f(x)\to+\infty$ as $x\to 1$ from below or above, $f(x)\to 1$ as $x\to\pm\infty$ and the curve has a minimum turning point $\left(-1,\frac{1}{2}\right)$.
On separate diagrams, sketch the following curves showing any important features :

(i) $y=f(|x-1|)$. 1

(ii) $y=\ln f(x)$. 1

(iii) $y=\frac{1}{f(x)}$. 2

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Question 15 (15 marks) **Use a separate writing booklet.** **Marks**

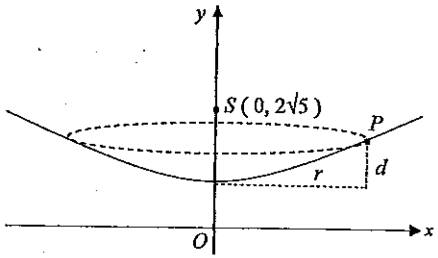
(a)(i) Using an appropriate substitution, prove that $\int_0^{2a} f(x)\,dx = \int_{-a}^{a} f(a-x)\,dx$ for $a > 0$. **1**

(ii) Hence show $\int_0^1 \sqrt{x(1-x)}\,dx = \frac{\pi}{8}$. **2**

(iii) If $I_n = \int_0^1 \sqrt{x}(1-x)^{\frac{n}{2}}\,dx$, $n = 0, 1, 2, \dots$, show that $I_n = \frac{n}{n+3} I_{n-2}$ for $n \ge 2$. **3**

(iv) Hence evaluate $\int_0^1 \sqrt{x}(1-x)^{\frac{5}{2}}\,dx$. **1**

(b)



The diagram shows an arc of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ cut off by the latus rectum through the focus $S(0, 2\sqrt{5})$. The hyperbola has asymptotes $y = \pm\frac{1}{2}x$.
This hyperbolic arc is rotated through one revolution about the y axis to form a bowl. P is a particle of mass m travelling around the inside of the bowl in a horizontal circle of radius r at a height d above the lowest point on the bowl with constant angular velocity ω .

(i) Show that $a = 4$ and $b = 2$. **2**

(ii) Show that if the normal to the hyperbola at P makes an angle θ with the vertical then $\tan\theta = \frac{r}{4(d+2)}$. **2**

(iii) If P has no tendency to slip on the surface, show that $\omega = \frac{1}{2}\sqrt{\frac{g}{d+2}}$ where g is the acceleration due to gravity. **2**

(iv) For this value of ω, find the force the particle exerts on the surface in terms of m, g and d. **2**

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Question 16 (15 marks) **Use a separate writing booklet.** **Marks**

(a) A particle P of mass m kg is projected vertically upwards in a medium where the resistance to motion has magnitude $\frac{1}{20}mv^2$ when the speed of the particle is v ms^{-1}. The terminal velocity of the particle is V ms^{-1}, and the speed of projection is λV ms^{-1} for some constant $\lambda > 0$.

(i) If during the upward journey the height of P above the point of projection is x metres, explain why $\ddot{x} = -\frac{1}{20}(V^2 + v^2)$. **2**

(ii) If H metres is the greatest height reached by the particle, show that $H = 10\ln(1+\lambda^2)$. **3**

(iii) Find the value of λ if the particle has 80% of its terminal velocity on return to its point of projection. **3**

(iv) For this value of λ, find in simplest exact form the time for the particle to fall from its highest point to the point of projection if $g = 10$ ms^{-2}. **2**

(b) ΔABC has sides a, b and c opposite angles A, B and C respectively. The perimeter of the triangle is P.

(i) If $c^2 = k(a+b)^2 + (1-k)(a-b)^2$, find k in terms of $\cos C$. **2**

(ii) Deduce that $P \ge (a+b)\sin\frac{1}{2}C + (b+c)\sin\frac{1}{2}A + (c+a)\sin\frac{1}{2}B$. **2**

(iii) What is the nature of the triangle if equality holds in this inequality for P ? Justify your answer. **1**

Section 1 Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1	C	$\lim_{x\to1}\frac{1-x}{1-\sqrt{x}}=\lim_{x\to1}\frac{(1-\sqrt{x})(1+\sqrt{x})}{1-\sqrt{x}}=\lim_{x\to1}(1+\sqrt{x})=2 \quad \therefore k=2$	H5
2	B	$(\sec x+\tan x)(\sec x-\tan x)=\sec^2 x-\tan^2 x=1$ $\quad (1)-(2)\Rightarrow 2\tan x=-\frac{3}{2}$ $\sec x+\tan x=\frac{1}{2}\ (1)\Rightarrow \sec x-\tan x=2\ (2)$ $\quad \tan x=-\frac{3}{4}$	H5
3	D	$\tan\left(\frac{\pi}{4}+\tan^{-1}x\right)=\frac{1+x}{1-x} \quad \therefore \tan^{-1}\left(\frac{1+x}{1-x}\right)=k+\tan^{-1}x,\ \begin{cases} k=\frac{\pi}{4}, & x<1 \\ k=\frac{\pi}{4}-\pi, & x>1\end{cases}$ $\therefore \frac{d}{dx}\tan^{-1}\left(\frac{x+1}{x-1}\right)=\frac{d}{dx}\left\{-\tan^{-1}\left(\frac{1+x}{1-x}\right)\right\}=-\frac{d}{dx}\left(k+\tan^{-1}x\right)=\frac{-1}{1+x^2}$	HE4
4	B	With 0, 1, 2, 3 A's respectively : $3!+3\times3!+3\times3+1=34$ arrangements	PE3
5	B	$\alpha(1+i)=1-i \quad \alpha(1+i)(1-i)=(1-i)^2 \quad \therefore 2\alpha=-2i \quad \therefore \alpha=-i$	E3
6	C	$P(x)=10(2x^6-3x^5+3x-2)\Rightarrow P(1)=0$ $\quad$ But $P'''(x)=600(4x^3-3x^2)$ $P'(x)=30(4x^5-5x^4+1) \Rightarrow P'(1)=0$ $\quad \therefore \quad P'''(1)\neq0$ $P''(x)=600(x^4-x^3) \Rightarrow P''(1)=0$ $\quad \therefore$ root 1 has multiplicity 3	E4
7	A	$x^2y-xy^2=6$ $\left(2xy+x^2\frac{dy}{dx}\right)-\left(1.y^2+2xy\frac{dy}{dx}\right)=0$ $\therefore \frac{dy}{dx}(x^2-2xy)-(y^2-2xy)=0$ $\therefore \frac{dy}{dx}=\frac{y^2-2xy}{x^2-2xy}$	E6
8	A	$k^2-1=k^2(1-e^2) \quad 1-\frac{1}{k^2}=1-e^2 \quad \therefore e^2=\frac{1}{k^2} \quad \therefore e=\frac{1}{k}$	E4
9	D	$t=\tan\frac{x}{2}$ $dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$ $\frac{2}{1+t^2}dt=dx$ $1+\cos x+\sin x=\frac{1+t^2+1-t^2+2t}{1+t^2}=\frac{2(1+t)}{1+t^2}$ $\int\frac{1}{1+\cos x+\sin x}dx=\int\frac{1+t^2}{2(1+t)}\cdot\frac{2}{1+t^2}dt=\int\frac{1}{(1+t)}dt$	E8
10	D	$A=\frac{1}{2}.2y.2y \Rightarrow \delta V=2(1-x^2)\delta x \quad \therefore V=\lim_{\delta x\to0}\sum_{x=-1}^{x=1}\delta V=2\int_{-1}^{1}(1-x^2)dx$	E7

Section II

Question 11

a. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • evaluates given expression	1
ii • finds reciprocal	1

Answer

i. $z=1+2i \Rightarrow iz+\bar{z}=i-2+1-2i=-1-i$

ii. $\frac{1}{z}=\frac{\bar{z}}{z\bar{z}}=\frac{\bar{z}}{|z|^2}=\frac{1-2i}{1^2+2^2}=\frac{1}{5}+\left(-\frac{2}{5}\right)i$

b. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
• writes transformed cubic equation, or finds sums of products taken 1, 2, 3 at a time	1
• forms required cubic equation in expanded form	1

Answer

Clearly $\alpha+1,\ \beta+1,\ \gamma+1$ are roots of $(x-1)^3+2(x-1)+1=0$. Rearranging gives $x^3-3x^2+5x-2=0$.

c. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• uses appropriate trig identities to simplify the integrand	1
• finds the primitive function	1

Answer

$$\int\frac{\cos 2x}{\cos^2 x}dx=\int\frac{2\cos^2 x-1}{\cos^2 x}dx=\int\left(2-\frac{1}{\cos^2 x}\right)dx=\int\left(2-\sec^2 x\right)dx=2x-\tan x+c$$

d. Outcomes assessed: E3, E4

Marking Guidelines

Criteria	Marks
i • explains why the sum is zero	1
ii • expands and rearranges to show required result	1
iii • relates $\cos\frac{2\pi}{5}$ to the roots of the quadratic equation $x^2+x-1=0$	1
• finds the positive root of this equation	1

Answer

i. $z^5-1=0$ has zero coefficient of z^4, hence the sum of the roots of the equation is zero.

ii. $\left(\omega+\frac{1}{\omega}\right)^2+\left(\omega+\frac{1}{\omega}\right)-1=\omega^2+\frac{1}{\omega^2}+2+\omega+\frac{1}{\omega}-1$

$=\frac{1}{\omega^2}\left(\omega^4+1+2\omega^2+\omega^3+\omega-\omega^2\right)$

$=\frac{1}{\omega^2}\left(1+\omega+\omega^2+\omega^3+\omega^4\right)$

$=0$

iii. $\omega+\frac{1}{\omega}=2\cos\frac{2\pi}{5}$

$\therefore 2\cos\frac{2\pi}{5}$ is the positive root of $x^2+x-1=0$

Then $x=\frac{-1\pm\sqrt{5}}{2} \Rightarrow \cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}$

Q11 (cont)

e. Outcomes assessed: E4, E6

Marking Guidelines

Criteria	Marks
i • writes the numerator in terms of $(x-2)$	1
• rearranges expression for y into required form	1
ii • differentiates to find the maximum turning point at the origin	1
• sketches curve with two branches, correct x intercepts and vertical asymptote	1
• shows oblique asymptote with both branches below this asymptote	1

Answer

i. $x^2(x-3)=\{(x-2)+2\}^2\{(x-2)-1\}$

$\therefore x^2(x-3)=\{(x-2)^2+4(x-2)+4\}^2\{(x-2)-1\}$

$=(x-2)^3+3(x-2)^2-4$

$=(x-2)^2(x+1)-4$

$\therefore y=\dfrac{x^2(x-3)}{(x-2)^2} \Rightarrow y=x+1-\dfrac{4}{(x-2)^2}$

ii. $y=x+1-4(x-2)^{-2}$

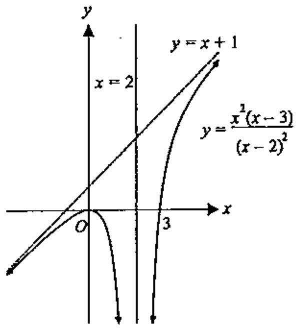
$\dfrac{dy}{dx}=1+\dfrac{8}{(x-2)^3}$ and $\dfrac{d^2y}{dx^2}=\dfrac{-24}{(x-2)^4}$

$\therefore$ curve has maximum turning point at $(0,0)$ and is concave down for all $x\neq 2$.

$\left|y-(x+1)\right|=\dfrac{4}{(x-2)^2}\to 0$ as $x\to\pm\infty$

hence $y=x+1$ is an oblique asymptote.

$y\to-\infty$ as $x\to 2$ $\therefore x=2$ is an asymptote.



Question 12

a. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • identifies the locus of P as a straight line and finds its gradient	1
• completes the equation of the locus	1
ii • uses coordinate geometry to find the minimum value of $\lvert z\rvert$	1

Answer

i. $|z-6|=|z+2i|$ Hence locus of P is the $\perp$ bisector of the join of $A(6,0)$ and $B(0,-2)$.

$m_{AB}=\frac{1}{3}$ and midpoint of AB has coordinates $(3,-1)$.

Hence locus of P is a line with gradient -3 and equation $y+1=-3(x-3)$, $3x+y-8=0$

ii. Minimum value of $|z|$ is $\perp$ distance d from $(0,0)$ to the line $3x+y-8=0$.

$d=\dfrac{|-8|}{\sqrt{3^2+1^2}}=\dfrac{8}{\sqrt{10}}$. Hence $|z|_{min}=\dfrac{4\sqrt{10}}{5}$

Q12 (cont)

b. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• converts the integral by substitution	1
• finds the primitive in terms of u	1
• finds the primitive in terms of x	1

Answer

$u=e^x+1$

$du=e^x dx$

$\displaystyle\int\frac{e^{2x}}{(e^x+1)^2}dx=\int\frac{e^x}{(e^x+1)^2}e^x dx$

$\displaystyle=\int\frac{u-1}{u^2}du$

$=\ln u+\dfrac{1}{u}+c$

$\displaystyle\therefore\int\frac{e^{2x}}{(e^x+1)^2}dx$

$=\ln(e^x+1)+\dfrac{1}{e^x+1}+c$

c. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • uses rotation clockwise by $\frac{\pi}{2}$ followed by enlargement by k is multiplication by $-ki$	1
• uses the relationships between the vectors to complete the proof	1
ii • writes the vector from M to Q in terms of w	1
• uses the moduli of complex numbers represented by the vectors to prove equal lengths	1

Answer

i. $\vec{OM}=\frac{1}{2}\vec{OP}=\frac{1}{2}\left\{\vec{OQ}+\vec{QP}\right\}=\frac{1}{2}\{w+k(-i)w\}=\frac{1}{2}(1-ki)w$. (Since $\angle OQP=\frac{\pi}{2}$ and $QP=k.OQ$)

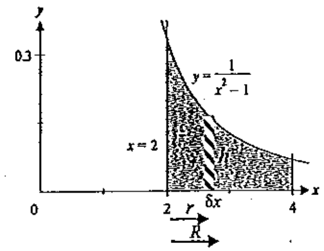
ii. $\vec{MQ}=\vec{OQ}-\vec{OM}=w-\frac{1}{2}(1-ki)w=\frac{1}{2}(1+ki)w$. $\therefore OM=\left|\frac{1}{2}(1-ki)w\right|=\frac{1}{2}\sqrt{1+k^2}\,|w|=\left|\frac{1}{2}(1+ki)w\right|=MQ$

d. Outcomes assessed: E7, E8

Marking Guidelines

Criteria	Marks
i • finds the volume of the cylindrical shell	1
• expresses the limiting sum of the cylindrical shells as a definite integral	1
ii • expresses the integrand as a sum of partial fractions	1
• finds the primitive	1
• evaluates in simplest exact form	1

Answer



i.

$R=x-2+\delta x$

$r=x-2$

$h=\dfrac{1}{x^2-1}$

$\delta V=\pi(R^2-r^2)h=\pi(R+r)(R-r)h$

$\therefore\delta V=\pi\{2(x-2)+\delta x\}.\delta x.h$

$\therefore$ ignoring second order terms

$\displaystyle V=\lim_{\delta x\to 0}\sum_{x=2}^{x=4}\pi\{2(x-2)\}.\frac{1}{x^2-1}.\delta x$

$\displaystyle=2\pi\int_2^4\frac{x-2}{x^2-1}dx$

Q12 d (cont)

ii.

$$2\pi\int_2^4 \frac{x-2}{x^2-1}\,dx = \pi\int_2^4\left(\frac{3}{x+1}-\frac{1}{x-1}\right)dx$$
$$=\pi\left[3\ln(x+1)-\ln(x-1)\right]_2^4$$

$$\therefore V=\pi\{3(\ln 5-\ln 3)-(\ln 3-\ln 1)\}$$
$$=\pi\ln\frac{125}{81}$$

Question 13

a. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• applies the substitution	1
• simplifies	1
• uses properties of the definite integral to deduce the value must be zero	1

Answer

$u=\frac{1}{x}$

$du=-\frac{1}{x^2}\,dx$

$-\frac{1}{u^2}du=dx$

$x=\frac{1}{e}\Rightarrow u=e$

$x=e\Rightarrow u=\frac{1}{e}$

$1+x=1+\frac{1}{u}=\frac{u+1}{u}$

$$\int_{\frac{1}{e}}^{e}\frac{\log_e x}{(1+x)^2}\,dx=\int_e^{\frac{1}{e}}-\log_e u\cdot\frac{u^2}{(1+u)^2}\cdot-\frac{1}{u^2}\,du$$
$$=-\int_{\frac{1}{e}}^{e}\frac{\log_e u}{(1+u)^2}\,du$$
$$=-\int_{\frac{1}{e}}^{e}\frac{\log_e x}{(1+x)^2}\,dx$$
$$\therefore\int_{\frac{1}{e}}^{e}\frac{\log_e x}{(1+x)^2}\,dx=0$$

b. Outcomes assessed: PE3, E2

Marking Guidelines

Criteria	Marks
i • expands the square of $(a+b+c)$	1
• uses the relationship between the sum of squares of a and b and the product $a\,b$	1
ii • considers appropriate replacements for a, b, c to deduce the result	1
iii • uses parts i. and ii. together to deduce the result	1

Answer

i. $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$

$2(a+b+c)^2=(a^2+b^2)+(b^2+c^2)+(c^2+a^2)+4ab+4bc+4ca$

$=(a-b)^2+(b-c)^2+(c-a)^2+6ab+6bc+6ca$

$\geq 6(ab+bc+ca)$ since $(a-b)^2\geq 0,\ (b-c)^2\geq 0,\ (c-a)^2\geq 0$

$\therefore(a+b+c)^2\geq 3(ab+bc+ca)$

ii. $a\to ab,\ b\to bc,\ c\to ca\ \Rightarrow\ (ab+bc+ca)^2\geq 3\{(ab)(bc)+(bc)(ca)+(ca)(ab)\}=3abc(a+b+c)$

iii. $(a+b+c)^4\geq 9(ab+bc+ca)^2\geq 27abc(a+b+c)$. $\quad\therefore(a+b+c)^3\geq 27abc$

Q13 (cont)

c. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • finds gradient of tangent and chord	1
• uses the common ratio property of a GP to deduce result	1
ii • writes the coordinates of M in terms of p and q	1
• uses the relationship between pq and k to deduce the locus	1

Answer

i. $m_{PQ}=\dfrac{c\left(\frac{1}{p}-\frac{1}{q}\right)}{c(p-q)}=\dfrac{q-p}{pq(p-q)}=-\dfrac{1}{pq}$

$x=ct \qquad y=\frac{c}{t}$

$\frac{dx}{dt}=c \qquad \frac{dy}{dt}=-\frac{c}{t^2}$

$\therefore\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{1}{t^2}$

$\therefore$ tangent at K has gradient $-\frac{1}{k^2}$

p,k,q in GP $\Rightarrow\frac{k}{p}=\frac{q}{k}$

$\therefore\frac{1}{pq}=\frac{1}{k^2}$

$\therefore PQ\parallel$ tangent at K.

ii. M has coordinates $\left(\frac{1}{2}c(p+q)\ ,\ \frac{\frac{1}{2}c(p+q)}{pq}\right)$. At M, $\frac{y}{x}=\frac{1}{pq}=\frac{1}{k^2}$. Locus of M is the line $y=\frac{x}{k^2}$.

d. Outcomes assessed: HE2, E9

Marking Guidelines

Criteria	Marks
• defines an appropriate sequence of statements and verifies that the first two are true	1
• uses the given recurrence relation to write expression for D_{r+1} conditional on truth of $D_n, n\leq r$	1
• shows understanding of sigma and factorial notation to begin simplification of this expression	1
• completes simplification into required form and finishes the induction process	1

Answer

Let $S(n)$, $n=1,2,3,\ldots$ be the sequence of statements defined by $S(n):\ D_n=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$.

Consider $S(n),\ n\leq 2$: $\ 1!\sum_{k=0}^{1}\frac{(-1)^k}{k!}=1!\left(\frac{(-1)^0}{0!}+\frac{(-1)^1}{1!}\right)=1(1-1)=0=D_1$

$2!\sum_{k=0}^{2}\frac{(-1)^k}{k!}=2!\left(1-1+\frac{(-1)^2}{2!}\right)=2\times\frac{1}{2}=1=D_2 \qquad \therefore S(n)$ *is true for* $n\leq 2$

If $S(n)$ *is true for* $n\leq r$: $\quad D_n=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}\quad$ *for* $\quad n\leq r \qquad$ **

Consider $S(r+1)$ *where* $r\geq 2$: $\ D_{r+1}=r(D_r+D_{r-1})=r\left\{r!\sum_{k=0}^{r}\frac{(-1)^k}{k!}+(r-1)!\sum_{k=0}^{r-1}\frac{(-1)^k}{k!}\right\}\quad$ *if* $S(n)$ *is true for* $n\leq r$ (**)

Q13 d (cont)

$$\text{Then } D_{r+1}=r\left\{(r-1)!\left[r\sum_{k=0}^{r-1}\frac{(-1)^k}{k!}+r\frac{(-1)^r}{r!}+\sum_{k=0}^{r-1}\frac{(-1)^k}{k!}\right]\right\}$$

$$=r!\left\{(r+1)\sum_{k=0}^{r-1}\frac{(-1)^k}{k!}+r\frac{(-1)^r}{r!}\right\}$$

$$=(r+1)!\sum_{k=0}^{r-1}\frac{(-1)^k}{k!}+r(-1)^r$$

$$=(r+1)!\left\{\sum_{k=0}^{r-1}\frac{(-1)^k}{k!}+\left[(r+1)-1\right]\frac{(-1)^r}{(r+1)!}\right\}$$

$$=(r+1)!\left\{\sum_{k=0}^{r-1}\frac{(-1)^k}{k!}+\frac{(-1)^r}{r!}+\frac{(-1)^{r+1}}{(r+1)!}\right\}$$

$$=(r+1)!\sum_{k=0}^{r+1}\frac{(-1)^k}{k!}$$

$\therefore$ if $S(n)$ is true for $n\le r$, where $r\ge 2$, then $S(r+1)$ is true. But $S(n)$ is true for $n\le 2$.

Hence by Mathematical Induction, $S(n)$ is true for all integers $n\ge 1$.

Question 14

a. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
• uses the fact that each of α,β,γ satisfies the equation to write an equation for the sum of cubes	1
• deduces the sum of the roots is zero to simplify this equation	1
• uses the value of the product of the roots in terms of q to deduce the required result	1

Answer

$\alpha,\ \beta,\ \gamma$ *roots of* $x^3+px+q=0 \Rightarrow$ $\alpha^3+p\alpha+q=0$, $\beta^3+p\beta+q=0$, $\gamma^3+p\gamma+q=0$

$\therefore\left(\alpha^3+\beta^3+\gamma^3\right)+p\left(\alpha+\beta+\gamma\right)+3q=0$

$\left(\alpha^3+\beta^3+\gamma^3\right)+p(0)+3q=0$

$\therefore \alpha^3+\beta^3+\gamma^3=3(-q)=3\alpha\beta\gamma$

b. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • recognises binomial probability distribution and calculates the probability as a fraction	1
ii • uses the symmetry of the situation to deduce the result	1
iii • recognises this event as the complement of the event *all odd*	1
• calculates the probability	1

Answer

i. $P(5\text{ even, }5\text{ odd})={}^{10}C_5\left(\tfrac{1}{2}\right)^5\left(\tfrac{1}{2}\right)^5=\tfrac{252}{1024}=\tfrac{63}{256}$

ii. $P(\#\text{ even}>\#\text{ odd})=P(\#\text{ odd}>\#\text{ even})=\tfrac{1}{2}\left\{1-P(\text{equal numbers of evens and odds})\right\}=\tfrac{1}{2}\left(1-\tfrac{63}{256}\right)=\tfrac{193}{512}$

iii. $P(\text{product is even})=P(\text{at least one is even})=1-P(\text{all odd})=1-\left(\tfrac{1}{2}\right)^{10}=\tfrac{1023}{1024}$

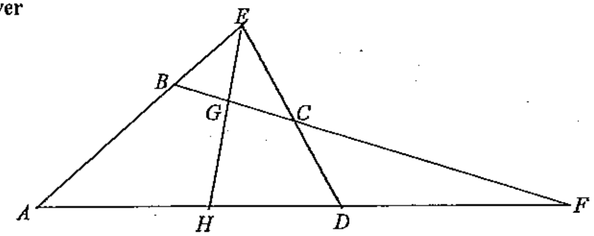
Q14 (cont)

c. Outcomes assessed: PE3, E2

Marking Guidelines

Criteria	Marks
• uses triangle exterior angle theorem for $\angle FGH$, $\angle FHG$ as sums of angles in ΔGEC, ΔHEA	1
• uses angle property of cyclic quadrilateral	1
• uses equal angles associated with given angle bisector	1
• deduces equal sides opposite equal angles in ΔFGH	1

Answer



$\angle FGH=\angle ECG+\angle CEG$ (*exterior* $\angle$ *of* ΔGEC *is equal to sum of interior opp.* $\angle$*'s*)

$=\angle EAH+\angle CEG$ (*exterior* $\angle$ *of cyclic quad. ABCD is equal to interior opp.* $\angle$)

$=\angle EAH+\angle AEH$ (*given EGH bisects* $\angle AED$)

$=\angle FHG$. (*exterior* $\angle$ *of* ΔHEA *is equal to sum of interior opp.* $\angle$*'s*)

$\therefore$ *In* ΔFGH, $FG=FH$ (*sides opp. equal* $\angle$*'s are equal*)

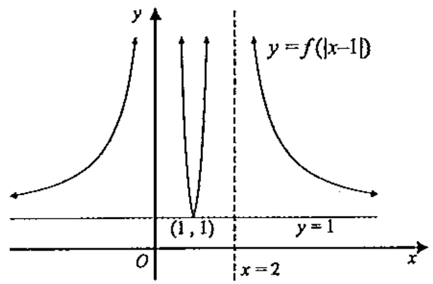
d. Outcomes assessed: E6

Marking Guidelines

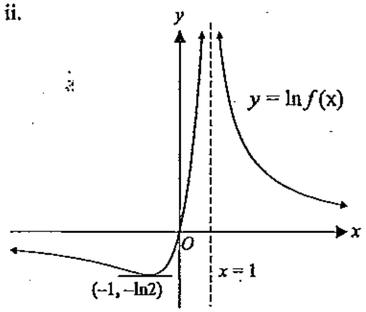
Criteria	Marks
i • reflects part of curve for $x\ge 0$ in y axis then translates 1 unit to the right	1
ii • shows curve with correct shape, turning point and asymptotes	1
iii • shows curve with correct shape, turning point and asymptote	1
• shows discontinuity at (1, 0)	1

Answer

i.

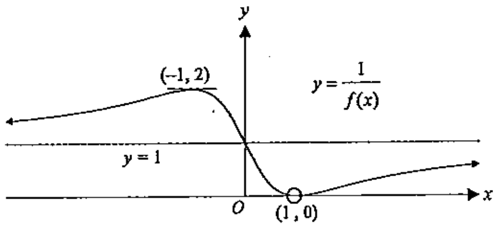


ii.



Q14 d (cont)

iii.



Question 15

a. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
i • applies an appropriate substitution to deduce the result	1
ii • applies the identity and rearranges	1
• recognises the semi-circle area to complete the evaluation	1
iii • applies integration by parts	1
• simplifies, then rearranges the resulting integrand into an appropriate form	1
• identifies the component integrals and completes the algebraic manipulation	1
iv • uses the recurrence relation and the result from ii. to evaluate the definite integral	1

Answer

i. $x = a - u$, $dx = -du$

$x=0 \Rightarrow u=a$

$x=2a \Rightarrow u=-a$

$$\int_0^{2a} f(x)\,dx = \int_a^{-a} f(a-u).-du = \int_{-a}^{a} f(a-u)\,du = \int_{-a}^{a} f(a-x)\,dx$$

ii. $$\int_0^1 \sqrt{x(1-x)}\,dx = \int_{-\frac12}^{\frac12} \sqrt{\left(\tfrac12 - x\right)\left(\tfrac12 + x\right)}\,dx = \int_{-\frac12}^{\frac12}\sqrt{\left(\tfrac14 - x^2\right)}\,dx = \tfrac12\pi\left(\tfrac12\right)^2 = \tfrac{\pi}{8}$$

iii. $$I_n = \int_0^1 \sqrt{x}(1-x)^{\frac n2}\,dx$$

$$= \left[\tfrac23 x^{\frac32}(1-x)^{\frac n2}\right]_0^1 - \tfrac23\cdot\tfrac n2\int_0^1 -x\sqrt{x}(1-x)^{\frac n2 - 1}\,dx \quad for\ n\ge 2$$

$$= 0 - \tfrac n3\int_0^1 \{(1-x)-1\}\sqrt{x}(1-x)^{\frac n2 - 1}\,dx$$

$$= -\tfrac n3 \int_0^1 \left\{\sqrt{x}(1-x)^{\frac n2} - \sqrt{x}(1-x)^{\frac{n-2}{2}}\right\}dx$$

$$3I_n = -n(I_n - I_{n-2})$$

$$(n+3)I_n = n\,I_{n-2}$$

$$I_n = \frac{n}{n+3}I_{n-2} \quad for\ \ n\ge 2$$

iv. $I_5 = \frac58 I_3 = \frac58\cdot\frac36 I_1$

But $I_1 = \int_0^1 \sqrt{x(1-x)}\,dx = \frac{\pi}{8}$

$\therefore I_5 = \frac{5\pi}{128}$

Q15 (cont)

b. Outcomes assessed: E3, E5

Marking Guidelines

Criteria	Marks
i • uses the equations of the asymptotes to find the eccentricity e	1
• uses the coordinates of the focus to find b and then a	1
ii • differentiates to find gradient of tangent to curve at P	1
• realises θ is also angle between tangent and horizontal to obtain expression for $\tan\theta$	1
iii • writes simultaneous equations from Newton's 2nd law, horizontal/vertical force resolution	1
• uses expression for $\tan\theta$ to obtain required expression for ω	1
iv • finds expression relating N, m, g, r, d using one of the Newton's law equations	1
• uses the equation of the hyperbola to substitute for r to obtain required expression for N	1

Answer

i. Asymptotes $y = \pm\frac{b}{a}x \Rightarrow \frac{b}{a} = \frac12$

$$a^2 = b^2(e^2 - 1)$$

$$\therefore e^2 = 1 + \left(\tfrac{a}{b}\right)^2 = 5$$

Then focus $(0, be) \Rightarrow be = 2\sqrt5$

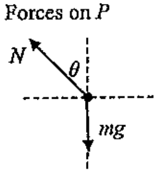
$\therefore b = 2,\ a = 4$

ii. $$\frac{y^2}{4} - \frac{x^2}{16} = 1 \Rightarrow \frac{2y}{4}\frac{dy}{dx} - \frac{2x}{16} = 0$$

$$\therefore \frac{dy}{dx} = \frac{x}{8}\cdot\frac{2}{y} = \frac{x}{4y} = \frac{r}{4(d+b)} = \frac{r}{4(d+2)} \quad at\ \ P$$

$\therefore \tan\theta = \frac{r}{4(d+2)}$, since θ is also angle between tangent and horizontal at P.

iii. Forces on P



Applying Newton's second law and resolving vertically and horizontally :

$$N\cos\theta = mg$$
$$N\sin\theta = mr\omega^2$$
$$\therefore\ \tan\theta = \frac{r\omega^2}{g}$$

$$\therefore \frac{r}{4(d+2)} = \frac{r\omega^2}{g}$$

$$\therefore \quad \omega = \tfrac12\sqrt{\frac{g}{d+2}}$$

iv. $$N^2 = (mg)^2\sec^2\theta$$

$$N^2 = (mg)^2(1+\tan^2\theta) = (mg)^2\left\{1 + \frac{r^2}{16(d+2)^2}\right\} = \left(\frac{mg}{d+2}\right)^2\left\{(d+2)^2 + \frac{r^2}{16}\right\}$$

$P(r,\ d+2)$ lies on the hyperbola

$$\therefore \frac{(d+2)^2}{4} - \frac{r^2}{16} = 1$$

$$(d+2)^2 + \frac{r^2}{16} = (d+2)^2 + \frac{(d+2)^2}{4} - 1 = \frac{5(d+2)^2 - 4}{4}$$

$$\therefore N^2 = \left\{\frac{mg}{2(d+2)}\right\}^2\left\{5(d+2)^2 - 4\right\}$$

$$N = \frac{mg\sqrt{5(d+2)^2 - 4}}{2(d+2)}$$

Applying Newton's 3rd Law, this is also the magnitude of the force the particle exerts on the surface.

Question 16

a. Outcomes assessed: E5

Marking Guidelines

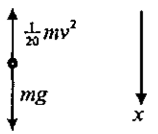
Criteria	Marks
i • considers forces and equation of motion during downward journey to relate V and g	1
• applies Newton's 2nd Law to obtain required equation of motion for upward journey	1
ii • obtains either $\frac{dx}{dv}$ or $\frac{dx}{d(v^2)}$ as a function of v	1
• finds x as a function of v, or evaluates a definite integral between appropriate limits	1
• obtains the required expression for H by substitution and simplification	1
iii • obtains $\frac{dx}{dv}$ or $\frac{dx}{d(v^2)}$ as a function of v from the equation of motion for downward journey	1
• finds x as a function of v, or evaluates a definite integral between appropriate limits	1
• substitutes value for H in terms of λ to determine value of λ	1
iv • finds t as a function of v by integration, or evaluates an appropriate definite integral	1
• substitutes values to obtain exact numerical expression for the time taken	1

Answer

i.

Downward journey

Forces on P / Using Newton's 2nd law



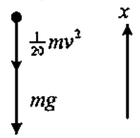
$\ddot{x} = g - \frac{1}{20}v^2$

$\ddot{x} \to 0 \ \text{as} \ \frac{1}{20}v^2 \to g$

$\therefore \frac{1}{20}V^2 = g$

Upward Journey

Forces on P / Using Newton's 2nd law



$\ddot{x} = -\left(g + \frac{1}{20}v^2\right)$

$\ddot{x} = -\frac{1}{20}\left(V^2 + v^2\right)$

ii. $v\frac{dv}{dx} = -\frac{1}{20}\left(V^2+v^2\right)$, $\begin{cases} x=0 \\ v=\lambda V \end{cases}$, $\begin{cases} x=H \\ v=0 \end{cases}$

$$-\frac{1}{20}\frac{dx}{dv} = \frac{v}{V^2+v^2}$$

$$-\frac{1}{10}\int_0^H dx = \int_{\lambda V}^0 \frac{2v}{V^2+v^2}\,dv$$

$$-\frac{1}{10}H = \Big[\ln\left(V^2+v^2\right)\Big]_{\lambda V}^0$$

$$= \ln V^2 - \ln\left(1+\lambda^2\right)V^2$$

$$= -\ln\left\{\frac{\left(1+\lambda^2\right)V^2}{V^2}\right\}$$

$$H = 10\ln\left(1+\lambda^2\right)$$

iii. $\ddot{x} = \frac{1}{20}\left(V^2 - v^2\right)$, $\begin{cases} x=0 \\ v=0 \end{cases}$, $\begin{cases} x=H \\ v=\frac{4}{5}V \end{cases}$

$$v\frac{dv}{dx} = \frac{1}{20}\left(V^2-v^2\right)$$

$$\frac{1}{20}\frac{dx}{dv} = \frac{v}{\left(V^2-v^2\right)}$$

$$\frac{1}{10}\int_0^H dx = \int_0^{\frac{4}{5}V} \frac{2v}{\left(V^2-v^2\right)}\,dv$$

$$\frac{1}{10}H = -\Big[\ln\left(V^2-v^2\right)\Big]_0^{\frac{4}{5}V}$$

$$= -\ln\tfrac{9}{25}V^2 + \ln V^2$$

$$\ln\left(1+\lambda^2\right) = \ln\tfrac{25}{9}$$

$$\therefore \lambda^2 = \tfrac{25}{9} - 1 = \tfrac{16}{9} \qquad \therefore \lambda = \tfrac{4}{3}$$

iv. $\frac{dv}{dt} = \frac{1}{20}\left(V^2 - v^2\right)$, $\begin{cases} t=0 \\ v=0 \end{cases}$, $\begin{cases} t=T \\ v=\frac{4}{5}V \end{cases}$

If P falls from highest point to point of projection in T s

$$\frac{1}{20}\frac{dt}{dv} = \frac{1}{V^2-v^2}$$

$$\frac{1}{10}V\frac{dt}{dv} = \frac{1}{V-v} + \frac{1}{V+v}$$

$$\frac{1}{10}V\int_0^T dt = \int_0^{\frac{4}{5}V}\left(\frac{1}{V+v} + \frac{1}{V-v}\right)dv$$

$$\frac{1}{10}VT = \Big[\ln\left(V+v\right) - \ln\left(V-v\right)\Big]_0^{\frac{4}{5}V}$$

$$\frac{1}{10}VT = \ln\tfrac{9}{5}V - \ln\tfrac{1}{5}V = \ln 9$$

$$\therefore T = \tfrac{10}{V}.2\ln 3 = \sqrt{2}\ln 3$$

(since $\frac{1}{100}V^2 = \frac{1}{5}g = 2$)

b. Outcomes assessed: H5, PE3, E2

Marking Guidelines

Criteria	Marks
i • compares the given expression for c^2 with that obtained by application of the cosine rule	1
• deduces k in terms of $\cos C$ from this comparison	1
ii • writes k in terms of $\sin\frac{1}{2}C$ and deduces $0<k<1$ to obtain inequality $c \ge (a+b)\sin\frac{1}{2}C$	1
• uses similar inequalities for a, b to deduce result for P	1
iii • explains why the triangle must be equilateral for equality to hold	1

Answer

i. Applying the cosine rule, $c^2 = a^2 + b^2 - 2ab\cos C$

$$a^2+b^2-2ab\cos C = k\left(a+b\right)^2 + \left(1-k\right)\left(a-b\right)^2$$

$$= \left(a^2+b^2\right)\left\{k+\left(1-k\right)\right\} + 2ab\left\{k-\left(1-k\right)\right\}$$

$$= a^2+b^2-2ab\left(1-2k\right)$$

$$\therefore 1-2k = \cos C \qquad \therefore k = \tfrac{1}{2}\left(1-\cos C\right)$$

ii.

$c^2 = k\left(a+b\right)^2 + \left(1-k\right)\left(a-b\right)^2$ *where* $k = \frac{1}{2}\left(1-\cos C\right) = \sin^2\frac{1}{2}C \Rightarrow 0<k<1$

$\therefore c^2 \ge k\left(a+b\right)^2$ *with equality if and only if* $a=b$, *since* $\left(a-b\right)^2 \ge 0$

$\therefore c \ge \left(a+b\right)\sin\frac{1}{2}C$

Using similar inequalities for a and b,

$$P = a+b+c \ge \left(a+b\right)\sin\tfrac{1}{2}C + \left(b+c\right)\sin\tfrac{1}{2}A + \left(c+a\right)\sin\tfrac{1}{2}B$$

iii. Equality holds in this inequality for P if and only if $\left(a-b\right)^2 = \left(b-c\right)^2 = \left(c-a\right)^2 = 0$, ie. $a=b=c$ so that the triangle is equilateral