NSW INDEPENDENT SCHOOLS

2016
Higher School Certificate
Trial Examination

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided at the back of the paper
- All necessary working should be shown in Question 11 – 16
- Write your student number and/or name at the top of every page

Total marks – 100

Section I - Pages 2 – 4

10 marks

Attempt Questions 1 - 10

Allow about 15 minutes for this section

Section II - Pages 5 – 10

90 marks

Attempt Questions 11 – 16

Allow about 2 hours 45 minutes for this section

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME:.................................

Marks

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

1 What is the value of $9 \cdot 8\times10^7 - 2 \cdot 3\times10^4$? 1

(A) $7 \cdot 5\times10^3$
(B) $7 \cdot 5\times10^4$
(C) $9 \cdot 7977\times10^6$
(D) $9 \cdot 7977\times10^7$

2 What is the value of $\sum_{k=1}^{4}(-1)^k k^2$? 1

(A) -30
(B) -10
(C) 10
(D) 30

3 Which of the following quadratic expressions is positive definite ? 1

(A) x^2+5x+2
(B) x^2+5x+4
(C) x^2+5x+6
(D) x^2+5x+8

4 Which of the following trigonometric expressions is equivalent to $\tan\left(\frac{\pi}{2}-x\right)$? 1

(A) $\tan x$
(B) $\cot x$
(C) $-\tan x$
(D) $-\cot x$

5 What is the range of the function $f(x)=\sqrt{1-x^2}$? 1

(A) $0<y<1$
(B) $0\le y\le1$
(C) $-1<y<1$
(D) $-1\le y\le1$

Marks

6 If k denotes the score when a fair die is rolled once, what is the probability that $(8k+1)$ is a perfect square ? 1

(A) $\frac{1}{6}$

(B) $\frac{1}{3}$

(C) $\frac{1}{2}$

(D) $\frac{2}{3}$

7 What is the value of $\int_{-2}^{2}|x|\,dx$? 1

(A) 0

(B) 2

(C) 4

(D) 8

8 What are the amplitude and period of the function $f(x)=2-\sin 2x$? 1

(A) Amplitude 1, period π

(B) Amplitude 1, period 2π

(C) Amplitude 2, period π

(D) Amplitude 2, period 2π

9 Which of the following is an expression for $\frac{d}{dx}\left(e^{2x}\tan x\right)$? 1

(A) $2e^{2x}\tan x$

(B) $e^{2x}\sec^2 x$

(C) $2\left(1+\tan^2 x\right)e^{2x}$

(D) $\left(1+\tan x\right)^2 e^{2x}$

10 What is the number of real roots of the equation $x(x-2)\log_e x=0$? 1

(A) 0

(B) 1

(C) 2

(D) 3

Marks

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer the questions in writing booklets provided. Use a separate writing booklet for each question. In Questions 11-16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet.**

(a) Find in simplest exact form the value of $\left(3-2\sqrt{2}\right)^2$. 2

(b) Solve the quadratic equation $2x^2-5x-3=0$. 2

(c) Differentiate with respect to x

(i) $\sin(1-x)$ 1

(ii) $x\log_e x$ 1

(d) Find the equation of the tangent to the curve $y=\frac{x}{x-1}$ at the point $(2,2)$ on the curve. 3

(e) Evaluate in simplest exact form $\int_1^2 \frac{x+1}{x}\,dx$. 3

(f) The region bounded by the curve $y=\frac{1}{2x+1}$ and the x axis between $x=0$ and $x=1$ is rotated about the x axis to form a solid. Find the volume of the solid of revolution in simplest exact form. 3

Marks

Question 12 (15 marks) **Use a separate writing booklet.**

(a) Find $\int \sec x(\cos x+\sec x)\,dx$. 2

(b) Find the coordinates of the point of inflexion on the curve $y=x^3-3x^2+6x$. 3

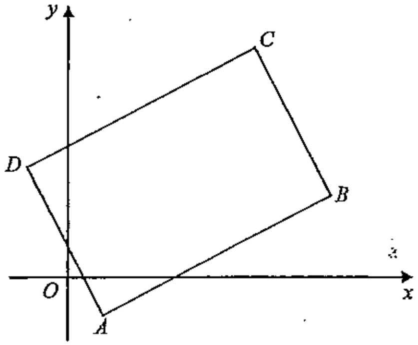
(c) A parabola has equation $8y=x^2-6x+1$.

(i) Write the equation in the form $(x-h)^2=4a(y-k)$. 1

(ii) Find the vertex and focal length of the parabola. 2

(iii) Find the coordinates of the focus and the equation of the directrix of the parabola. 2

(d) In the diagram below, $ABCD$ is a parallelogram. Side AB has equation $x-2y-3=0$, side BC has equation $2x+y-16=0$ and vertex D has coordinates $(-1,3)$.



NOT TO SCALE

(i) Show that $ABCD$ is a rectangle. 2

(ii) Find the coordinates of the vertex C. 3

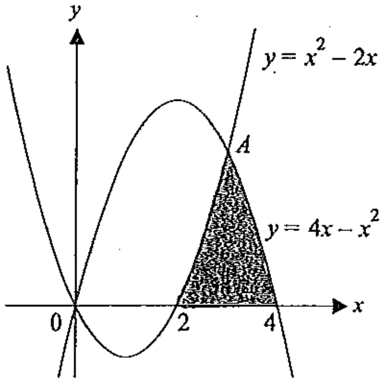
Marks

Question 13 (15 marks) **Use a separate writing booklet.**

(a) The gradient function of a curve $y=f(x)$ is given by $f'(x)=\frac{x}{2}+\frac{4}{\sqrt{x}}$. 3

The curve passes through the point $(4,5)$. Find the equation of the curve.

(b) The diagram below shows the parabolas $y=4x-x^2$ and $y=x^2-2x$. The graphs intersect at the origin O and the point A.



(i) Find the x coordinate of the point A. 1

(ii) Find the area of the shaded region bounded by the two parabolas and the x axis. 3

(c) Blaise rolls a fair die twice.

(i) Find as a fraction the probability that the scores include a 3 or a 5 but not both. 2

(ii) Find as a fraction the probability that both scores are odd given that the highest score showing is a 5. 2

(d) Use Simpson's Rule with 5 function values to approximate $\int_1^9 (\log_e x)^2\,dx$, giving your answer correct to 2 significant figures. 4

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Marks

Question 14 (15 marks) **Use a separate writing booklet.**

(a) Find the values of x for which the function $y = x - x^2$ is decreasing. **2**

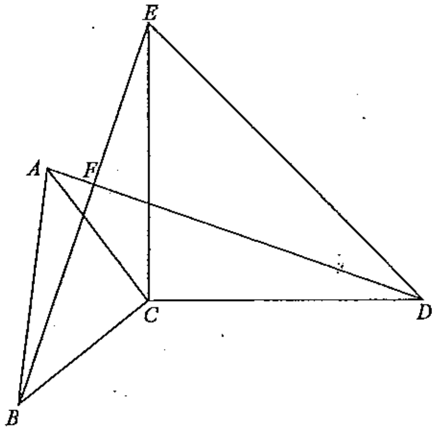
(b) The 8th term of an Arithmetic Progression is 23 and the 11th term is four times the 3rd term. Find the first term and the common difference. **3**

(c) After time t years the value $\$V$ of a car is given by $V = 24\,000\,e^{-0.1t}$.

(i) Find the decrease in the value of the car during the fourth year. **2**

(ii) Find the percentage decrease in the value of the car during the fourth year. **2**

(d) In ΔABC, $AC = BC$ and $\angle BCA = 90°$. In ΔCDE, $DC = EC$ and $\angle ECD = 90°$. DA and BE intersect at F.



(i) Copy the diagram. Prove that $\Delta BCE \equiv \Delta ACD$. **3**

(ii) Show that $DA \perp BE$. **3**

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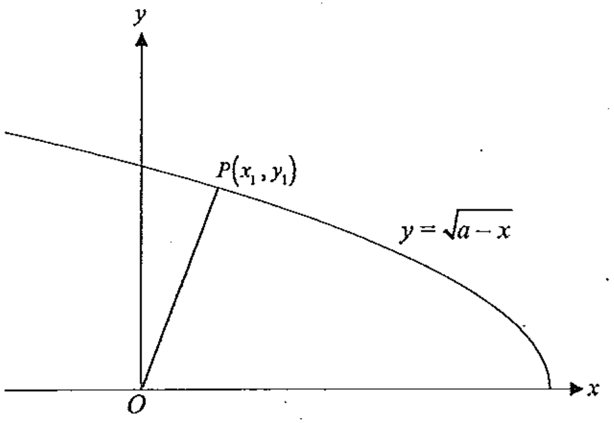
Marks

Question 15 (15 marks) **Use a separate writing booklet.**

(a) Find in simplest exact form the value of $\int_0^{\log_e 3} e^{2x}\,dx$. **2**

(b) Solve the equation $2\log_2 x - \log_2(x+4) = 1$. **3**

(c) The diagram shows the graph of the curve $y = \sqrt{a-x}$ where $a > 0$. The normal to the curve at the point $P(x_1, y_1)$ passes through the origin $O(0,0)$. By considering the gradient of OP in two different ways, find the value of x_1 . **3**



(d)(i) Solve the equation $\sin x = \cos x$ for $0 \le x \le 2\pi$. **2**

(ii) On the same diagram, sketch the graphs of the curves $y = \cos x$ and $y = \sin x$ for $0 \le x \le 2\pi$, showing clearly the intercepts on the coordinate axes. **2**

(iii) Find in simplest exact form the area of the region enclosed by the curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le 2\pi$. **3**

Marks

Question 16 (15 marks) **Use a separate writing booklet.**

(a) Given that the limiting sums S_1 and S_2 of the series both exist, where

$S_1 = 1 + \sin^2 x + \sin^4 x + \sin^6 x + \ldots$

$S_2 = 1 + \cos^2 x + \cos^4 x + \cos^6 x + \ldots$

(i) Show that $S_1 = \sec^2 x$ and $S_2 = \operatorname{cosec}^2 x$. **2**

(ii) Show that $S_1 + S_2 = S_1 S_2$. **2**

(b) A particle is moving in a straight line. After time t seconds its displacement x metres from a fixed point O on the line is given by $x = t - 3\log_e(t+1)$. The particle returns to its starting point after T seconds.

(i) Find when the particle is at rest. **1**

(ii) Find in simplest exact form the distance travelled by the particle in the first T seconds of its motion. **2**

(iii) Show that $e^T = (T+1)^3$. **2**

(c) A cylindrical container closed at both ends is to be made from thin sheet metal. The container is to have a radius of r cm and a height of h cm such that its volume is $2000\pi \text{ cm}^3$.
(Such a cylinder closed at both ends has surface area S and volume V given by the formulae $S = 2\pi r^2 + 2\pi rh$ and $V = \pi r^2 h$.)

(i) Show that the area of sheet metal required to make the container is $\left(2\pi r^2 + \frac{4000\pi}{r}\right) \text{cm}^2$. **2**

(ii) Hence find the minimum area of sheet metal required to make the container. **4**

Section 1 **Questions 1-10 (1 mark each)**

Question	Answer	Solution	Outcomes
1	D	$9{\cdot}8\times10^7 - 2{\cdot}3\times10^4 = (9{\cdot}8 - 0{\cdot}0023)\times10^7 = 9{\cdot}7977\times10^7$	P3
2	C	$\sum_{k=1}^{4}(-1)^k k^2 = -1+4-9+16=10$	H5
3	D	For x^2+5x+8, $\Delta = 25-32<0$. For other expressions, $\Delta > 0$	P4
4	B	$\tan\left(\frac{\pi}{2}-x\right)=\cot x$	H5
5	B	$0 \le \sqrt{1-x^2} \le 1 \quad \therefore 0 \le f(x) \le 1$	P5
6	C	$k=1,2,\ldots,6 \Rightarrow 8k+1=\underline{9},17,\underline{25},33,41,\underline{49} \quad \therefore P(square)=\frac{3}{6}=\frac{1}{2}$	H5
7	C	$\int_{-2}^{2}\lvert x\rvert\,dx = 2\int_0^2 x\,dx = \left[x^2\right]_0^2 = 4$	H8
8	A	$-1 \le \sin 2x \le 1 \quad \therefore 1 \le 2-\sin 2x \le 3 \quad \therefore$ Amplitude is 1. Period is $\frac{2\pi}{2}=\pi$	H5
9	D	$\frac{d}{dx}\left(e^{2x}\tan x\right) = 2e^{2x}\tan x + e^{2x}\sec^2 x = e^{2x}\left(2\tan x + \tan^2 x + 1\right) = \left(1+\tan x\right)^2 e^{2x}$	H5
10	C	$x(x-2)\log_e x = 0$ for $x=2,1$ ($x=0$ is outside the domain) $\therefore$ 2 real roots	H3

Section II

Question 11

a. Outcomes assessed: P3

Marking Guidelines

Criteria	Marks
• expands the square	1
• simplifies	1

Answer

$\left(3-2\sqrt{2}\right)^2 = 9-12\sqrt{2}+8 = 17-12\sqrt{2}$

b. Outcomes assessed: P4

Marking Guidelines

Criteria	Marks
• factorises, completes the square or substitutes into the quadratic formula	1
• writes down both solutions	1

Answer

$2x^2-5x-3=0 \qquad \therefore 2x+1=0 \text{ or } x-3=0$

$(2x+1)(x-3)=0 \qquad x=-\frac{1}{2} \text{ or } x=3$

Q1 (cont)

c. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
i • applies the chain rule	1
ii • applies the product rule	1

Answer

i. $\frac{d}{dx}\sin(1-x)=-\cos(1-x)$ ii. $\frac{d}{dx}x\log_e x=1.\log_e x+x.\frac{1}{x}=\log_e x+1$

d. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• differentiates	1
• finds the gradient of the tangent	1
• finds the equation of the tangent	1

Answer

$$y=\frac{x}{x-1}$$

$$\frac{dy}{dx}=\frac{1.(x-1)-x.1}{(x-1)^2}$$

$$=\frac{-1}{(x-1)^2}$$

$\therefore$ tangent at $(2,2)$ has gradient -1

and equation $y-2=-1(x-2)$

$$y-2=-x+2$$

$$y=-x+4$$

e. Outcomes assessed: H8

Marking Guidelines

Criteria	Marks
• finds the primitive function	1
• evaluates	1
• simplifies	1

Answer

$$\int_1^2\frac{x+1}{x}\,dx=\int_1^2\left(1+\frac{1}{x}\right)dx=\left[x+\ln x\right]_1^2=(2-1)+(\ln 2-\ln 1)=1+\ln 2$$

f. Outcomes assessed: H8

Marking Guidelines

Criteria	Marks
• expresses the volume as a definite integral	1
• finds the primitive function	1
• evaluates	1

Answer

$$V=\pi\int_0^1\frac{1}{(2x+1)^2}\,dx=-\frac{\pi}{2}\left[\frac{1}{2x+1}\right]_0^1=-\tfrac{\pi}{2}\left(\tfrac{1}{3}-1\right)=\tfrac{\pi}{3}$$

$\therefore$ volume is $\frac{\pi}{3}$ cu. units

Question 12

a. Outcomes assessed: H8

Marking Guidelines

Criteria	Marks
• simplifies the integrand	1
• finds the primitive function	1

Answer

$$\int\sec x(\cos x+\sec x)\,dx=\int\left(1+\sec^2 x\right)dx=x+\tan x+c$$

b. Outcomes assessed: H6

Marking Guidelines

Criteria	Marks
• finds the second derivative	1
• finds the zero of the second derivative and the corresponding *y* value	1
• checks the change of sign of the second derivative near its zero to verify inflexion point	1

Answer

$$y=x^3-3x^2+6x$$

$$\frac{dy}{dx}=3x^2-6x+6$$

$$\frac{d^2y}{dx^2}=6x-6$$

$\therefore\frac{d^2y}{dx^2}=0$ at $(1,4)$

Sign of $\frac{d^2y}{dx^2}$: $\overset{-ve}{\text{——}}\underset{1}{\overset{0}{\text{—}}}\overset{+ve}{\text{——}}\xrightarrow[x]{}$

$\therefore(1,4)$ is a point of inflexion.

c. Outcomes assessed: P4

Marking Guidelines

Criteria	Marks
i • writes equation in required form	1
ii • states coordinates of vertex	1
• states focal length	1
iii • states coordinates of focus	1
• states equation of directrix	1

Answer

i.

$$8y=x^2-6x+1$$

$$8y+8=x^2-6x+9$$

$$8(y+1)=(x-3)^2$$

$$(x-3)^2=4\times 2\{y-(-1)\}$$

ii. Vertex $(3,-1)$, focal length 2

iii. Focus $(3,\ 1)$, directrix $y=-3$

Q12(cont)

d. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
i • finds the gradients of *AB* and *BC* and shows product is –1	1
• deduces *ABCD* is a rectangle, giving a reason	1
ii • finds equation of *CD*	1
• solves simultaneous equations for coordinates of *C*, finding one coordinate	1
• finds the second coordinate of *C*	1

Answer

i. $AB: \quad x-2y-3=0$ $\qquad BC: \quad 2x+y-16=0$

$x-3=2y$ $\qquad y=-2x+16$ $\qquad \therefore m_{AB}.m_{BC}=-1$ and $\angle ABC=90°$

$\frac{1}{2}x-\frac{3}{2}=y$ $\qquad \therefore ABCD$ is a rectangle

$\therefore m_{AB}=\frac{1}{2}$ $\qquad \therefore m_{BC}=-2$ $\qquad$ *(parallelogram with one vertex ∠ 90°)*

ii. $CD \parallel AB \quad \therefore CD$ has equation $x-2y=k$ for some constant k.

$D(-1,3)$ on line $\Rightarrow -1-6=k \quad \therefore k=-7$

Hence at C $\quad x-2y=-7 \quad (1)$ $\qquad (2)-2\times(1) \Rightarrow 5y=30$ $\qquad y=6 \Rightarrow x=2y-7=5$

$2x+y=16 \quad (2)$ $\qquad y=6$ $\qquad \therefore C$ has coordinates $(5,6)$

Question 13

a. Outcomes assessed: H8

Marking Guidelines

Criteria	Marks
• finds a primitive of the first term of the gradient function	1
• finds a primitive of the second term of the gradient function	1
• evaluates the constant to find the equation of the curve	1

Answer

$f'(x)=\frac{1}{2}x+4x^{-\frac{1}{2}}$

$f(x)=\frac{1}{4}x^2+8x^{\frac{1}{2}}+c$

$=\frac{1}{4}x^2+8\sqrt{x}+c$

$f(4)=5 \Rightarrow 5=\frac{1}{4}\times16+8\sqrt{4}+c$

$5=20+c$

$\therefore c=-15$

$\therefore$ equation of curve is

$y=\frac{x^2}{4}+8\sqrt{x}-15$

b. Outcomes assessed: P4, H8

Marking Guidelines

Criteria	Marks
i • finds *x* coordinate of *A*	1
ii • writes definite integrals for required area	1
• finds the primitive functions	1
• evaluates	1

Answer

i. At A, $\quad x^2-2x=4x-x^2$

$2x^2-6x=0$

$2x(x-3)=0$

$\therefore x\neq0 \Rightarrow x=3$

ii. Area is given by $\int_2^3(x^2-2x)dx+\int_3^4(4x-x^2)dx$

$=\left[\frac{1}{3}x^3-x^2\right]_2^3+\left[2x^2-\frac{1}{3}x^3\right]_3^4$

$=\frac{1}{3}(27-8)-(9-4)+2(16-9)-\frac{1}{3}(64-27)$

Area is 3 sq.units.

Q13(cont)

c. Outcomes assessed: H5, H9

Marking Guidelines

Criteria	Marks
i • counts outcomes in required event	1
• writes probability as a fraction	1
ii • identifies reduced sample space and counts possible outcomes	1
• counts outcomes in required event and writes probability as a fraction	1

Answer

i.

1st roll

2nd roll

	1	2	3	4	5	6
1			*		*	
2			*		*	
3	*	*	*	*		*
4			*		*	
5	*	*		*	*	*
6			*		*	

Sample space has 36 equally likely outcomes.
Required event has 18 outcomes *
Probability is $\frac{18}{36}=\frac{1}{2}$.

ii.

1st roll

2nd roll

	1	2	3	4	5	6
1					√*	
2					√	
3					√*	
4					√	
5	√*	√	√*	√	√*	
6						

Sample space has 9 equally likely outcomes √.
Required event has 5 outcomes *
Probability is $\frac{5}{9}$

d. Outcomes assessed: H8

Marking Guidelines

Criteria	Marks
• uses correct *x* values and value of *h*	1
• substitutes correctly into formula	1
• calculates correctly	1
• makes no intermediate rounding errors and expresses approximation to 2 significant figures	1

Answer

x	1	3	5	7	9
$f(x)$	0	$(\ln3)^2$	$(\ln5)^2$	$(\ln7)^2$	$(\ln9)^2$
×	1	4	2	4	1

$h=2$

$$\int_1^9(\log_e x)^2\,dx \approx \frac{2}{3}\left\{0+4\times(\ln3)^2+2\times(\ln5)^2+4\times(\ln7)^2+(\ln9)^2\right\}$$

$$\approx 19{\cdot}988$$

$$\approx 20 \quad \textit{(to 2 sig. fig.)}$$

Question 14

a. Outcomes assessed: H6

Marking Guidelines

Criteria	Marks
• forms inequality with first derivative negative	1
• solves for x	1

Answer

$y=x-x^2 \Rightarrow \dfrac{dy}{dx}=1-2x$ $\quad 1-2x<0$ $\quad 1<2x$ $\quad \therefore$ decreasing for $x>\frac{1}{2}$

b. Outcomes assessed: P4, H5

Marking Guidelines

Criteria	Marks
• writes a pair of simultaneous equations from the given information	1
• uses elimination or substitution to write an equation in one pronumeral	1
• solves and finds both required values	1

Answer

$T_8=23 \Rightarrow a+7d=23$ $\qquad 3a+21d=69 \quad (1)$ $\qquad (1)-(2) \Rightarrow 23d=69$

$T_{11}=4T_3 \Rightarrow a+10d=4(a+2d)$ $\qquad 3a-2d=0 \quad (2)$ $\qquad \therefore d=3$ and $a=2$

First term 2, common difference 3.

c. Outcomes assessed: H1, H3

Marking Guidelines

Criteria	Marks
i • writes numerical expression for decrease in value	1
• calculates this decrease	1
ii • writes numerical expression for percentage decrease	1
• calculates this percentage decrease	1

Answer

i. $\Delta V=24\,000e^{-0\cdot1\times4}-24\,000e^{-0\cdot1\times3}=24\,000\left(e^{-0\cdot4}-e^{-0\cdot3}\right)\approx-1691\cdot956$ *Ans.* \$1692 (nearest \$)

ii. $\dfrac{24\,000\left(e^{-0\cdot4}-e^{-0\cdot3}\right)}{24\,000e^{-0\cdot3}}\times100=\left(e^{-0\cdot1}-1\right)\times100\approx-9\cdot516$ *Ans.* 9·5% (to 1 decimal place)

Q14(cont)

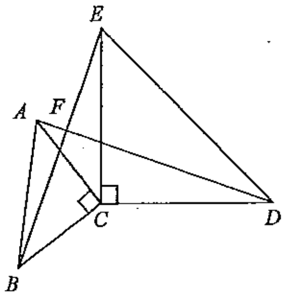
d. Outcomes assessed: H2, H5

Marking Guidelines

Criteria	Marks
i • applies an appropriate test for congruence, presenting conventional deductive proof	1
• explains why angles at C are equal	1
• notes given equalities of sides	1
ii • explains why triangle CDE has 45° angles at E and D	1
• writes $\angle FED$, $\angle EDF$ in terms of $\angle BEC$, $\angle ADC$ respectively	1
• uses the congruence from i. to show $\angle EFD=90°$	1

Answer

In the diagram $DC=EC$ and $AC=BC$.



i. In ΔBCE , ΔACD

$BC=AC$ (*given*)

$EC=DC$ (*given*)

$\angle BCE=90°+\angle ACE$ (*adding adj* $\angle$*'s, given* $\angle BCA=90°$)

$\angle ACD=90°+\angle ACE$ (*adding adj* $\angle$*'s, given* $\angle ECD=90°$)

$\therefore \angle BCE=\angle ACD$

$\therefore \Delta BCE\equiv\Delta ACD$ (*SAS*)

ii. In isosceles right ΔECD , $\angle CDE=\angle CED=45°$
($\angle$ *sum is* 180°, $\angle$*'s opp. equal sides are equal*)

In ΔEFD ,

$\angle FED=\angle CED+\angle BEC=45°+\angle BEC$

$\angle EDF=\angle CDE-\angle ADC=45°-\angle ADC$

But $\angle BEC=\angle ADC$ (*corresp.* $\angle$*'s in congruent* Δ*'s are equal*)

$\therefore \angle FED+\angle EDF=90°$

$\therefore \angle EFD=90°$ ($\angle$ *sum of* Δ *is* 180°)

$\therefore DA\perp BE$

Question 15

a. Outcomes assessed: H3, H8

Marking Guidelines

Criteria	Marks
• finds the primitive and evaluates e^0	1
• completes the evaluation	1

Answer

$$\int_0^{\log_e 3} e^{2x}\,dx=\tfrac{1}{2}\left[e^{2x}\right]_0^{\ln 3}=\tfrac{1}{2}\left(e^{2\ln 3}-e^0\right)=\tfrac{1}{2}\left(e^{\ln 3^2}-1\right)=\tfrac{1}{2}\left(3^2-1\right)=4$$

Q15 (cont)

b. Outcomes assessed: P4, H3

Marking Guidelines

Criteria	Marks
• uses log laws to obtain quadratic equation in x • factorises this quadratic (or completes square or applies quadratic formula) • realises domain is restricted and quotes only one solution for x	1 1 1

Answer

$2\log_2 x - \log_2(x+4) = 1$ with domain $x > 0$

$2\log_2 x = \log_2 2 + \log_2(x+4)$

$\log_2 x^2 = \log_2 2(x+4)$

$\therefore\ x^2 = 2x+8$ *and* $x > 0$

$x^2 - 2x - 8 = 0$

$\therefore (x-4)(x+2) = 0$ *and* $x > 0$ $\quad \therefore x = 4$

c. Outcomes assessed: H5, H7

Marking Guidelines

Criteria	Marks
• uses differentiation to find the gradient of the normal *OP* • uses coordinate geometry to find the gradient of *OP* and hence writes equation for x_1 • solves the equation	1 1 1

Answer

$$y = (a-x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -\tfrac{1}{2}(a-x)^{-\frac{1}{2}}$$

$$= \frac{-1}{2\sqrt{a-x}}$$

$$\therefore m_{OP} = 2\sqrt{a-x_1}$$

$$m_{OP} = \frac{y_1}{x_1}$$

$$= \frac{\sqrt{a-x_1}}{x_1}$$

$$\therefore \quad 2\sqrt{a-x_1} = \frac{\sqrt{a-x_1}}{x_1}$$

$$2x_1\sqrt{a-x_1} = \sqrt{a-x_1}$$

$$\sqrt{a-x_1}\,(2x_1-1) = 0$$

$$\therefore x_1 = a \quad or \quad x_1 = \tfrac{1}{2}$$

d. Outcomes assessed: H5, H8

Marking Guidelines

Criteria	Marks
i • finds one solution, using an appropriate trigonometric identity • states the second solution ii • sketches one graph showing the intercepts on the axes • sketches the second graph showing the intercepts on the axes iii • expresses the required area in terms of a definite integral • finds the primitive • evaluates	1 1 1 1 1 1 1

Answer

i.

$\sin x = \cos x \quad 0 \le x \le 2\pi$

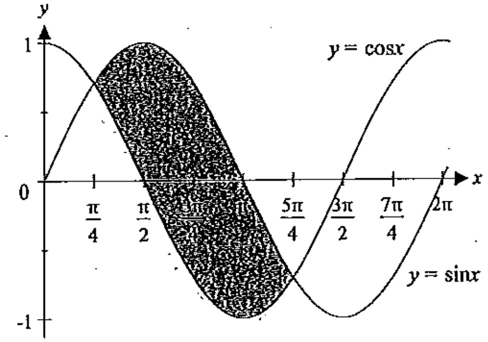
$\dfrac{\sin x}{\cos x} = 1$ (since $\cos x \ne 0$)

$\therefore \tan x = 1$

$x = \frac{\pi}{4}$, $\frac{5\pi}{4}$

Q15 d. (cont)

ii.



iii. The required area enclosed between the curves is given by

$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx$$

$$= -\left[\cos x + \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= -\left\{\left(\cos\tfrac{5\pi}{4} - \cos\tfrac{\pi}{4}\right) + \left(\sin\tfrac{5\pi}{4} - \sin\tfrac{\pi}{4}\right)\right\}$$

$$= -\left\{-\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}\right\}$$

$$= 2\sqrt{2}$$

Area is $2\sqrt{2}$ sq. units

Question 16

a. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
i • deduces the required expression for one limiting sum • deduces the required expression for the second limiting sum ii • takes a common denominator for the sum • uses an appropriate trig. identity to simplify the sum and show it reduces to the product	1 1 1 1

Answer

i. Given the limiting sums exists (*true for* $x \ne m\frac{\pi}{2}$ *for any integer* m)

$S_1 = 1 + \sin^2 x + \sin^4 x + \sin^6 x + \ldots$ is the limiting sum of the GP with $a = 1,\ r = \sin^2 x$

$S_2 = 1 + \cos^2 x + \cos^4 x + \cos^6 x + \ldots$ is the limiting sum of the GP with $a = 1,\ r = \cos^2 x$

$\therefore S_1 = \dfrac{1}{1-\sin^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x$ and $\therefore S_2 = \dfrac{1}{1-\cos^2 x} = \dfrac{1}{\sin^2 x} = \text{cosec}^2 x$

ii. $S_1 + S_2 = \dfrac{1}{\cos^2 x} + \dfrac{1}{\sin^2 x} = \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x.\sin^2 x} = \dfrac{1}{\cos^2 x.\sin^2 x} = S_1 S_2$

Q16(cont)

b. Outcomes assessed: H5, H3

Marking Guidelines

Criteria	Marks
i • solves $\dot{x}=0$	1
ii • finds x values for $t=0$ and $t=2$	1
• considers change of direction to find distance travelled	1
iii • writes an equation for T	1
• uses log laws to obtain required expression	1

Answer

i. $x=t-3\log_e(t+1)$

$\dot{x}=1-\dfrac{3}{t+1}$

$\dot{x}=0 \Rightarrow t=2$

Particle at rest at time 2 s.

ii. $\dot{x}<0 \;\; for \;\; t<2, \quad \dot{x}>0 \;\; for \;\; t>2$

Particle is initially at $x=0$, moves left to $x=-(3\ln 3-2)$ at time $t=2$, then changes direction returning to $x=0$ at time $t=T$.

Distance travelled in first T seconds is $2(3\ln 3-2)$ m.

iii. $0=T-3\log_e(T+1)$ $\quad \therefore T=3\log_e(T+1)$

$T=\log_e(T+1)^3$

$\therefore e^T=(T+1)^3$

c. Outcomes assessed: H1, H4, H5

Marking Guidelines

Criteria	Marks
i • uses fixed volume to express h in terms of r	1
• expresses surface area in terms of r only	1
ii • finds first derivative of S with respect to r	1
• finds the zero of the first derivative	1
• verifies that this zero corresponds to a minimum S value	1
• finds the minimum S value	1

Answer

i. $\pi r^2 h=2000\pi \qquad \therefore h=\dfrac{2000}{r^2} \qquad \therefore S=2\pi r^2+2\pi rh=2\pi r^2+\dfrac{4000\pi}{r}$

ii. $\dfrac{dS}{dr}=4\pi r-\dfrac{4000\pi}{r^2}$

$\therefore \dfrac{dS}{dr}=\dfrac{4\pi}{r^2}(r^3-1000)$

$\dfrac{dS}{dr}=0 \;\; for \;\; r=10$

$\dfrac{d^2S}{dr^2}=4\pi+\dfrac{8000\pi}{r^3}>0 \quad for \;\; r>0$

$\therefore$ Minimum S for $r=10$, $\;S_{\min}=2\pi\times 100+\dfrac{4000\pi}{10}$

$\therefore$ Minimum area is 600π cm^2