NSW INDEPENDENT SCHOOLS

2012
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in Question 11 – 14
- Write your student number and/or name at the top of every page

Total marks – 70

Section I - Pages 3 – 4

10 marks

Attempt Questions 1 - 10

Allow about 15 minutes for this section

Section II - Pages 5 – 9

60 marks

Attempt Questions 11 – 14

Allow about 1 hour 45 minutes for this section

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ……………………….

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

STUDENT NUMBER/NAME: ....

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Select the alternative A, B, C or D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	B	C	D
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

Student name / number ....

Marks

Section I

1. What is the value of $\lim_{x \to 0} \frac{\sin 3x}{4x}$? **1**

(A) 0
(B) $\frac{3}{4}$
(C) 1
(D) $\frac{4}{3}$

2. The equation $x^3 + 2x^2 - 3x - 6 = 0$ has roots $\alpha, -\alpha, \beta$. What is the value of β? **1**

(A) -6
(B) -2
(C) 2
(D) 6

3. If $y = e^{x^2}$, which of the following is an expression for $\frac{d^2y}{dx^2}$? **1**

(A) $2xe^{x^2}$
(B) $2e^{x^2}$
(C) $4x^2e^{x^2}$
(D) $(4x^2 + 2)e^{x^2}$

4. Which of the following is an expression for $\int \sin^2 2x \, dx$? **1**

(A) $\frac{1}{2}x - \frac{1}{8}\sin 4x + c$
(B) $\frac{1}{2}x - \frac{1}{4}\sin 4x + c$
(C) $\frac{1}{2}x + \frac{1}{8}\sin 4x + c$
(D) $\frac{1}{2}x + \frac{1}{4}\sin 4x + c$

5. What is the value of $\cos n\pi$ for $n = 1, 2, 3, \ldots$? **1**

(A) -1
(B) $(-1)^n$
(C) $(-1)^{n+1}$
(D) $\frac{(-1) + (-1)^n}{2}$

Student name / number ....

Marks

6. If $\frac{dN}{dt}=0\cdot1(N-100)$ and $N=300$ when $t=0$, which of the following is an expression for N? **1**

(A) $200+100e^{0\cdot1t}$
(B) $300+100e^{0\cdot1t}$
(C) $100+200e^{0\cdot1t}$
(D) $100+300e^{0\cdot1t}$

7. Which of the following represents the inverse of $f(x)=e^x-2$? **1**

(A) $f^{-1}(x)=\frac{1}{e^x-2}$
(B) $f^{-1}(x)=e^{-x}-\frac{1}{2}$
(C) $f^{-1}(x)=\log_e x+2$
(D) $f^{-1}(x)=\log_e(x+2)$

8. Which of the following is an expression for $\int\frac{2x}{\sqrt{1+x^2}}dx$? **1**

(A) $\log_e(1+x^2)+c$
(B) $\log_e\sqrt{1+x^2}+c$
(C) $\sqrt{1+x^2}+c$
(D) $2\sqrt{1+x^2}+c$

9. If $t=\tan\frac{x}{2}$ which of the following is an expression for $\frac{dx}{dt}$? **1**

(A) $\frac{1}{2}(1+t^2)$
(B) $1+t^2$
(C) $\frac{2}{1+t^2}$
(D) $\frac{1}{1+t^2}$

10. What is the value of the term independent of x in the expansion of $\left(x^2+\frac{2}{x}\right)^6$? **1**

(A) 60
(B) 160
(C) 192
(D) 240

Student name / number ....

Marks

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hour 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

All necessary working should be shown in every question.

Question 11 **Begin a new booklet**

(a) Find the number of ways in which 3 boys and 3 girls can be arranged in a straight line so that the girls are all next to each other, but the boys are not all next to each other. **2**

(b) Find the coordinates of the point $P(x, y)$ which divides the interval joining the points $A(-2,5)$ and $B(4,1)$ externally in the ratio 3 : 1. **2**

(c) Solve the inequality $\frac{1}{x-1}<1$. **2**

(d) Consider the function $f(x)=x+e^{-x}$.
(i) Find the coordinates and nature of the stationary point on the curve $y=f(x)$. **2**
(ii) Find the equation of the asymptote on the graph of the curve $y=f(x)$. **1**

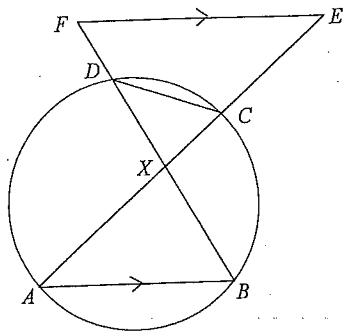
(e) $P(2at, at^2)$ is a point on the parabola $x^2=4ay$ with focus $F(0,a)$.
(i) Use differentiation to show that the normal to the parabola at P has gradient $-\frac{1}{t}$. **1**
(ii) If θ is the acute angle between the normal to the parabola at P and the line PF show that $\tan\theta=|t|$. **2**

Student name / number ....

Marks

Question 11 (cont)

(f)



AC and BD are two chords of a circle which intersect at point X inside the circle.
E is a point on AC produced and F is a point on BD produced such that $FE \parallel AB$.
Show that $DCEF$ is a cyclic quadrilateral. **3**

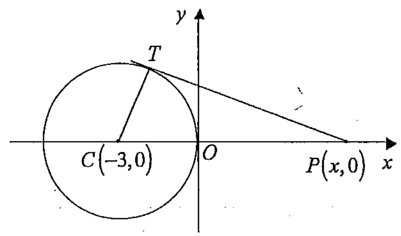
Student name / number ....

Marks

Question 12 **Begin a new booklet**

(a) Show that the equation $\log_e x + x = 0$ has a root between $x = 0\cdot5$ and $x = 1$. **2**

(b)



$P(x,0)$ is a point on the positive x-axis. T is the point of contact of a tangent drawn from P to the circle with centre $C(-3,0)$ and radius 3.

(i) Show that $PT = \sqrt{x^2+6x}$. **1**

(ii) If the units in the above diagram are cm, and P is moving along the x-axis away from O at a constant rate of $0\cdot1 \text{ cm s}^{-1}$, find the rate of change of PT when $x = 2$ cm. **2**

(c) Use the substitution $u = x+1$ to evaluate $\int_1^3 \frac{x+2}{(x+1)^2}\,dx$, giving the answer in simplest exact form. **3**

(d) Use Mathematical Induction to show that for all positive integers $n \ge 2$, **3**

$$2\times1 + 3\times2 + 4\times3 + \ldots + n(n-1) = \tfrac{1}{3}n(n^2-1) .$$

(e)(i) If $\sin^{-1}x - \cos^{-1}x = \frac{\pi}{3}$, show that $\sin^{-1}x = \frac{5\pi}{12}$. **2**

(ii) Hence solve $\sin^{-1}x - \cos^{-1}x = \frac{\pi}{3}$, giving the answer in simplest surd form. **2**

Student name / number ....

Marks

Question 13 **Begin a new booklet**

(a) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line, velocity $v\ \text{ms}^{-1}$ given by $v = \dfrac{2}{3\sqrt{x}}$ and acceleration $a\ \text{ms}^{-2}$. Initially the particle is 1 metre to the right of O.

(i) Show that $a = \dfrac{-2}{9x^2}$. **1**

(ii) Show that $x = (t+1)^{\frac{2}{3}}$. **2**

(iii) Show that the particle is always moving away from O and slowing down. **2**

(iv) Find the time taken for the speed of the particle to drop to 10% of its initial speed. **2**

(b) Consider the function $f(x) = 2\cos^{-1}(x-1)$ where $1 \le x \le 2$.

(i) Sketch the curve $y = f(x)$ showing clearly the coordinates of the endpoints. **2**

(ii) Find the equation of the inverse function $f^{-1}(x)$ and state its domain. **2**

(c) On any roll of a biased die there is a probability p of getting a 'six'. If the die is rolled 6 times the probability of getting at least 5 'sixes' is $\frac{1}{2}$.

(i) Show that $10p^6 - 12p^5 + 1 = 0$. **2**

(ii) Use one application of Newton's Method with an initial approximation of $p_0 = 0\cdot75$ to find the next approximation to the value of p, giving your answer correct to 2 decimal places. **2**

Student name / number ....

Marks

Question 14 **Begin a new booklet**

(a) A particle is moving in a straight line and performing Simple Harmonic Motion. At time t seconds it has displacement x metres from a fixed point O on the line, given by $x = 2\cos\left(2t - \frac{\pi}{4}\right)$, velocity $v\ \text{ms}^{-1}$ and acceleration $\ddot{x}\ \text{ms}^{-2}$.

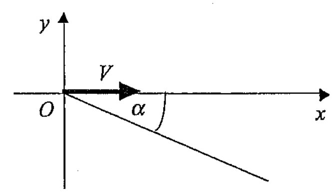
(i) Show that $v^2 - x\ddot{x} = 16$. **2**

(ii) Sketch the graph of x as a function of t for $0 \le t \le \pi$ showing clearly the coordinates of the endpoints. **2**

(iii) Show that the particle first returns to its starting point after one quarter of its period. **1**

(iv) Find the time taken by the particle to travel the first 100 metres of its motion. **2**

(b)



A particle is projected horizontally from a point O with speed $V\ \text{ms}^{-1}$ down a slope which is inclined at an angle $\alpha = \tan^{-1}\frac{1}{2}$ below the horizontal. The particle moves in a vertical plane under gravity where the acceleration due to gravity is $g\ \text{ms}^{-2}$. At time t seconds the horizontal and vertical displacements from O, x metres and y metres respectively, are given by $x = Vt$ and $y = -\frac{1}{2}gt^2$. (DO NOT PROVE THESE RESULTS.)

(i) Show the particle hits the slope after time $\dfrac{V}{g}$ seconds. **2**

(ii) Show that the particle hits the slope with velocity $V\sqrt{2}\ \text{ms}^{-1}$ at an angle of $45°$ to the vertical. **2**

(c)(i) Show that $\sum_{r=1}^{n}(1+x)^{r-1} = \sum_{r=1}^{n} {}^{n}C_{r}\, x^{r-1}$. **2**

(ii) Hence show that for $n \ge 3$, $\sum_{r=2}^{n-1} {}^{r}C_{2} = {}^{n}C_{3}$. **2**

Section I

Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1.	**B**	$\lim_{x\to0}\frac{\sin 3x}{4x}=\frac{3}{4}\left(\lim_{x\to0}\frac{\sin 3x}{3x}\right)=\frac{3}{4}\times1=\frac{3}{4}$	H5
2.	**B**	$\alpha+(-\alpha)+\beta=-2 \quad \therefore \beta=-2$	PE3
3.	**D**	$\frac{d}{dx}\left(e^{x^2}\right)=2xe^{x^2} \quad \therefore \frac{d^2}{dx^2}\left(e^{x^2}\right)=2e^{x^2}+2x.2xe^{x^2}=\left(4x^2+2\right)e^{x^2}$	PE5
4.	**A**	$\int \sin^2 2x\,dx=\int \tfrac{1}{2}(1-\cos 4x)\,dx=\tfrac{1}{2}x-\tfrac{1}{8}\sin 4x+c$	H5
5.	**B**	$\cos n\pi=\begin{cases}1, & n \text{ even}\\ -1, & n \text{ odd}\end{cases} \quad \therefore \cos n\pi=(-1)^n,\ \ n=1,2,3,\dots$	HE2
6.	**C**	$\frac{dN}{dt}=0\cdot1(N-100) \Rightarrow N-100=Ae^{0\cdot1t}$ for some constant A Then $N=300,\ t=0 \Rightarrow 200=A \quad \therefore N=100+200e^{0\cdot1t}$	HE3
7.	**D**	$y=e^x-2 \Leftrightarrow \log_e(y+2)=x \quad \therefore f^{-1}(x)=\log_e(x+2)$	HE4
8.	**D**	$\int\frac{2x}{\sqrt{1+x^2}}\,dx=\int(1+x^2)^{-\frac{1}{2}}.2x\,dx=2(1+x^2)^{\frac{1}{2}}+c=2\sqrt{1+x^2}+c$	HE6
9.	**C**	$t=\tan\frac{x}{2} \Rightarrow \frac{dt}{dx}=\frac{1}{2}\sec^2\frac{x}{2}=\frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) \quad \therefore \frac{dx}{dt}=\frac{2}{1+t^2}$	HE4
10.	**D**	$\left(x^2+\frac{2}{x}\right)^6$ has general term ${}^6C_r\left(\frac{2}{x}\right)^r\left(x^2\right)^{6-r}={}^6C_r2^rx^{12-3r},\ r=0,1,\dots,6$ $r=4$ gives constant term ${}^6C_4 2^4=240$	HE3

Section II

Question 11

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• counts arrangements for one suitable pattern of B's and G's	1
• adds arrangements for the second possible pattern	1

Answer

With restriction, pattern B G G G B B or B B G G G B

$3\times3!\times2! + 2!\times3!\times3 = 72$ arrangements.

b. Outcomes assessed : P4

Marking Guidelines

Criteria	Marks
• finds the x coordinate of P	1
• finds the y coordinate of P	1

Answer

$A(-2,5) \qquad B(4,1)$

$3 \ : \ -1$

$P\left(\frac{3\times4+(-1)\times(-2)}{3+(-1)},\ \frac{3\times1+(-1)\times5}{3+(-1)}\right) \qquad \therefore P(7,-1)$

c. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• obtains one inequality satisfied by x	1
• obtains the second inequality and indicates correctly how the two inequalities are combined	1

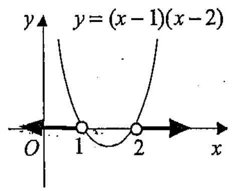
Answer

$\frac{1}{x-1}<1$

$x-1<(x-1)^2\ ,\ \ x\neq1$

$0<(x-1)\{(x-1)-1\}$

$\therefore\ 0<(x-1)(x-2),\ \ x\neq1$



$\therefore x<1 \ \ or \ \ x>2$

Question 11 (cont)

d. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
i. • considers the first derivative to find coordinates of the stationary point	1
• applies first or second derivative test to determine nature of the stationary point	1
ii. • writes the equation of the asymptote	1

Answer

i.

$f(x)=x+e^{-x}$ $\quad f'(x)=0 \Rightarrow e^{-x}=1$

$f'(x)=1-e^{-x}$ $\quad x=0$

$f''(x)=e^{-x}>0$ for all x $\quad$ where $f''(0)>0$

Hence there is one stationary point $(0,1)$ which is a minimum turning point.

ii. $y=x+e^{-x}$ $\quad \therefore y-x=e^{-x}\to 0$ as $x\to+\infty$. Hence the line $y=x$ is an asymptote as $x\to+\infty$.

e. Outcomes assessed : H5, PE3, PE4

Marking Guidelines

Criteria	Marks
i. • finds gradient of normal by differentiation	1
ii. • finds gradient of PF	1
• substitutes gradients into formula for $\tan\theta$ and simplifies to obtain required result	1

Answer

i.

$y=at^2 \qquad \frac{dy}{dt}=2at$

$x=2at \qquad \frac{dx}{dt}=2a$

$\therefore \frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=t$

Hence normal at $P(2at, at^2)$ has gradient $-\frac{1}{t}$.

ii.

$m_{PF}=\frac{a(t^2-1)}{2at}=\frac{(t^2-1)}{2t}$

$\therefore \tan\theta=\left|\frac{\frac{1}{2t}\{(t^2-1)-(-2)\}}{1+\frac{1}{2t}(t^2-1).(-\frac{1}{t})}\right|=\left|\frac{t(t^2+1)}{2t^2-(t^2-1)}\right|$

$\therefore \tan\theta=\left|\frac{t(t^2+1)}{t^2+1}\right|=|t|$

f. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
• quotes an appropriate test for a cyclic quadrilateral	1
• writes a sequence of deductions resulting in the application of this test	1
• quotes appropriate geometric properties to support these deductions	1

Answer

$\angle EFD=\angle DBA$ *(Alternate ∠'s between parallel lines are equal)*

$\angle DBA = DCA$ *(∠'s subtended at the circumference of the circle by the same arc DA are equal)*

$\therefore \angle DCA=\angle EFD$

$\therefore$ EFDC is a cyclic quadrilateral *(an exterior ∠ is equal to the opposite interior ∠)*

Question 12

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• shows $f(0\cdot5)$ and $f(1)$ have opposite signs	1
• notes the continuity of the function f	1

Answer

$f(x)=\log_e x+x$ is a continuous function for $x>0$ and $f(0\cdot5)\approx-0\cdot19<0$, $f(1)=1>0$

$\therefore f(x)=0$ for some x such that $0\cdot5<x<1$.

b. Outcomes assessed : P4, HE5

Marking Guidelines

Criteria	Marks
i. • shows required result	1
ii. • finds required rate of change in terms of x and $\frac{dx}{dt}$	1
• substitutes given values to calculate rate of change in cm s^{-1}	1

Answer

i. $PT^2=PC^2-TC^2$ (by Pythagoras' theorem, since tangent ⊥ radius drawn to point of contact)

$\therefore PT=\sqrt{(x+3)^2-3^2}=\sqrt{x^2+6x}$

ii. Let $s=PT$: $\quad s^2=x^2+6x$

$2s\frac{ds}{dt}=(2x+6)\frac{dx}{dt}$ $\qquad x=2,\ \frac{dx}{dt}=0\cdot1$

$\frac{ds}{dt}=\frac{x+3}{\sqrt{x^2+6x}}\frac{dx}{dt}$ $\qquad \therefore \frac{ds}{dt}=\frac{5}{4}\times0\cdot1$

$\therefore PT$ is increasing at $0\cdot125$ cm s^{-1}.

c. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• converts to definite integral in terms of u	1
• finds primitive	1
• substitutes limits and simplifies	1

Answer

$u=x+1$

$du=dx$

$x=1\Rightarrow u=2$

$x=3\Rightarrow u=4$

$I=\int_1^3\frac{x+2}{(x+1)^2}dx=\int_2^4\frac{u+1}{u^2}dx$

$\therefore I=\left[\ln u-\frac{1}{u}\right]_2^4=(\ln4-\ln2)-(\frac{1}{4}-\frac{1}{2})=\ln2+\frac{1}{4}$

Question 12 (cont)

d. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• verifies that the equality holds for $n=2$	1
• shows that the truth of the statement for $n=k$ implies the truth of the statement for $n=k+1$	1
• presents the proof in a way that shows an understanding of Mathematical Induction	1

Answer

Let $S(n)$, $n=2,3,4,\ldots$ be the sequence of statements defined by $S(n)$: $\sum_{r=2}^{n} r(r-1)=\frac{1}{3}n(n^2-1)$

Consider $S(2)$: $LHS=2\times 1=2$ $\quad RHS=\frac{1}{2}2(2^2-1)=2$ $\quad \therefore S(2)$ is true.

If $S(k)$ is true: $\sum_{r=2}^{k} r(r-1)=\frac{1}{3}k(k^2-1)$ **

Consider $S(k+1)$: $\sum_{r=2}^{k+1} r(r-1)=\sum_{r=2}^{k} r(r-1)+(k+1)k$

$=\frac{1}{3}k(k^2-1)+(k+1)k$ if $S(k)$ is true, using **

$=\frac{1}{3}(k+1)\{k(k-1)+3k\}$

$=\frac{1}{3}(k+1)\{(k+1)^2-1\}$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(2)$ is true. Hence $S(3)$ is true, and then $S(4)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers $n\geq 2$.

e. Outcomes assessed : H5, HE4

Marking Guidelines

Criteria	Marks
i. • uses the identity $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$	1
• obtains the required result	1
ii. • writes x as $\sin(\alpha\pm\beta)$ where the exact trig. ratios of α,β are known as surds	1
• writes x in simplest surd form	1

Answer

i. $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for all $-1\leq x\leq 1$.

$\therefore \sin^{-1}x-\cos^{-1}x=\frac{\pi}{3} \Rightarrow 2\sin^{-1}x=\frac{\pi}{2}+\frac{\pi}{3}$

$\sin^{-1}x=\frac{5\pi}{12}$

ii. $\sin^{-1}x=\frac{5\pi}{12}$

$x=\sin\frac{5\pi}{12}=\sin(\frac{\pi}{4}+\frac{\pi}{6})$

$x=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$

Question 13

a. Outcomes assessed : HE5

Marking Guidelines

Criteria	Marks
i. • derives $\frac{1}{2}v^2$ with respect to x to give a in terms of x	1
ii. • integrates to find t as a function of x	1
• uses the initial conditions to evaluate the constant of integration; writes x as a function of t	1
iii. • explains why particle is moving away from O	1
• explains why particle is slowing down	1
iv. • finds x when v has 10% of initial value	1
• finds t for this x value	1

Answer

i. $v^2=\frac{4}{9x}$ $\quad \therefore a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\frac{-2}{9x^2}$

ii. $\frac{dx}{dt}=\frac{2}{3\sqrt{x}}$

$\frac{dt}{dx}=\frac{3}{2}x^{\frac{1}{2}}$

$t=x^{\frac{3}{2}}+c$

$\left.\begin{array}{l}t=0\\x=1\end{array}\right]\Rightarrow$ $c=-1$

$t=x^{\frac{3}{2}}-1$

$x=(t+1)^{\frac{2}{3}}$

iii. $v=\frac{2}{3\sqrt{x}}>0$ and $a=\frac{-2}{9x^2}<0$ for all x, and initially $x=1$. Hence particle initially moves to the right, away from O, and continues to move right away from O while slowing down (since v decreases as x increases, or alternatively since v and a have opposite signs)

iv. $t=0 \Rightarrow v=\frac{2}{3}$

When $v=\frac{1}{10}\times\frac{2}{3}$, $x=\left(\frac{2}{3v}\right)^2=100$

$t=100^{\frac{3}{2}}-1=999$

$\therefore$ 10% of initial velocity after 999 s = 16 min 39 s.

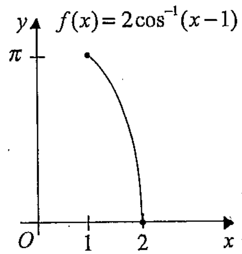
b. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
i. • shows curve of correct shape with x-intercept 2	1
• shows coordinates of endpoint at $x=1$	1
ii. • writes equation of inverse function	1
• states domain	1

Answer

i.



ii.

$y=2\cos^{-1}(x-1) \Leftrightarrow x=\cos(\frac{1}{2}y)+1,\ 0\leq y\leq\pi$

$\therefore f^{-1}(x)=\cos(\frac{1}{2}x)+1,\ 0\leq x\leq\pi$

Question 13 (cont)

c. **Outcomes assessed : PE3, HE3**

Marking Guidelines

Criteria	Marks
i. • writes an expression for the probability of at least 5 sixes in terms of p	1
• evaluates binomial coefficients, writes and simplifies equation for p	1
ii. • substitutes $p_0 = 0\cdot 75$ into formula for finding next approximation by Newton's Method	1
• calculates next approximation	1

Answer

i.

$$ {}^6C_5 p^5(1-p) + {}^6C_6 p^6 = \tfrac{1}{2} $$
$$ 2(6p^5 - 6p^6 + p^6) = 1 $$
$$ 10p^6 - 12p^5 + 1 = 0 $$

ii.

Let $f(p) = 10p^6 - 12p^5 + 1$ Then $f(0\cdot 75) \approx -0\cdot 0679$

$f'(p) = 60p^5 - 60p^4$ $f'(0\cdot 75) \approx -4.7461$

Next approximation is $p_1 = 0\cdot 75 - \dfrac{f(0\cdot 75)}{f'(0\cdot 75)} \approx 0\cdot 74$

Question 14

a. **Outcomes assessed : HE3**

Marking Guidelines

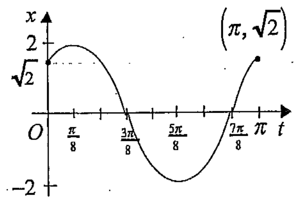
Criteria	Marks
i. • obtains $\dot{x}$, $\ddot{x}$ by differentiation	1
• uses trig. identity to establish result	1
ii. • sketches sinusoidal curve with correct amplitude and period	1
• shows appropriate lateral shift for the given phase angle with coordinates of endpoints	1
iii. • uses the symmetry of the graph to deduce the result	1
iv. • shows that 100 m of travel corresponds to $12\cdot 5$ complete oscillations	1
• deduces that time taken is $12\cdot 5$ periods	1

Answer

i.

$$ x = 2\cos\left(2t - \tfrac{\pi}{4}\right) $$
$$ \dot{x} = -4\sin\left(2t - \tfrac{\pi}{4}\right) $$
$$ \ddot{x} = -8\cos\left(2t - \tfrac{\pi}{4}\right) $$
$$ v^2 - x\,\ddot{x} = 16\left\{\sin^2\left(2t-\tfrac{\pi}{4}\right) + \cos^2\left(2t - \tfrac{\pi}{4}\right)\right\} $$
$$ = 16 $$

ii.



iii. Using the symmetry in the graph, first return to $x = \sqrt{2}$ is when $t = \frac{\pi}{4}$.

Period is $T = \frac{2\pi}{2} = \pi$ seconds. Hence first return to starting point is after $\frac{1}{4}T$.

iv. Particle travels 8m in one oscillation. $100\text{ m} = 12 \times 8\text{ m} + 4\text{ m}$

Time taken is $12 \times \pi + \frac{\pi}{2} = \frac{25\pi}{2}$ seconds

Question 14 (cont)

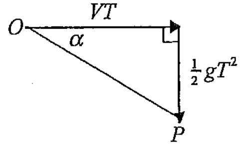
b. **Outcomes assessed : HE3**

Marking Guidelines

Criteria	Marks
i. • shows the displacement of the particle down the slope at impact as vector sum of x and y	1
• uses $\tan\alpha = \frac{1}{2}$ to deduce result	1
ii. • expresses $\dot{x}$ and $\dot{y}$ at time of impact in terms of V	1
• deduces direction and magnitude of velocity at impact from vector sum of $\dot{x}$ and $\dot{y}$	1

Answer

i. Let particle hit slope at time T seconds at point P.

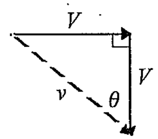


$$ \tan\alpha = \frac{\frac{1}{2}gT^2}{VT} = \frac{1}{2}\cdot\frac{gT}{V} $$

But $\tan\alpha = \frac{1}{2}$. $\therefore T = \dfrac{V}{g}$

ii. Let particle hit slope at angle θ to vertical with velocity v ms^{-1}

Then v has horizontal and vertical components $\dot{x} = V$, $\dot{y} = -gT = -V$



Hence $\theta = 45°$ (isosceles right Δ has equal $\angle$'s of $45°$)

and $v = V\sqrt{2}$ (applying Pythagoras' theorem)

c. **Outcomes assessed : H5, HE3**

Marking Guidelines

Criteria	Marks
i. • recognises sum of a GP to simplify LHS	1
• uses Binomial theorem to simplify RHS	1
ii. • chooses to equate coefficients of x^2 on both sides of identity in i., with correct RHS	1
• obtains coefficient of x^2 on LHS as a sum of binomial coefficients to deduce result	1

Answer

i. $\displaystyle\sum_{r=1}^{n}(1+x)^{r-1} = 1 + (1+x) + (1+x)^2 + \ldots + (1+x)^{n-1} = \frac{1.\{(1+x)^n - 1\}}{(1+x)-1} = \frac{1}{x}\{(1+x)^n - 1\}$ (using S_n for GP)

$\displaystyle\sum_{r=1}^{n} {}^nC_r\, x^{r-1} = \frac{1}{x}\left\{\sum_{r=0}^{n} {}^nC_r\, x^r - {}^nC_0\, x^0\right\} = \frac{1}{x}\{(1+x)^n - 1\}$ (using Binomial theorem)

Hence $\displaystyle\sum_{r=1}^{n}(1+x)^{r-1} = \sum_{r=1}^{n} {}^nC_r\, x^{r-1}$.

ii. For $n \geq 3$, equating coefficients of x^2 on both sides of the identity in (i):

$$ {}^2C_2 + {}^3C_2 + \ldots + {}^{n-1}C_2 = {}^nC_3 $$

Hence $\displaystyle\sum_{r=2}^{n-1} {}^rC_2 = {}^nC_3$