NSW INDEPENDENT SCHOOLS

2015
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided at the back of the paper
- All necessary working should be shown in Question 11 – 14
- Write your student number and/or name at the top of every page

Total marks – 70

Section I - Pages 2 – 5

10 marks

Attempt Questions 1 - 10

Allow about 15 minutes for this section

Section II - Pages 6 – 9

60 marks

Attempt Questions 11 – 14

Allow about 1 hour 45 minutes for this section

This paper MUST NOT be removed from the examination room.

STUDENT NUMBER/NAME: ………………………

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

Marks

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

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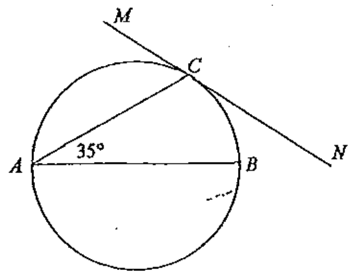
1 In the diagram, AB is a diameter of the circle and MCN is the tangent to the circle at C. $\angle CAB = 35°$. What is the size of $\angle MCA$? **1**



(A) 35°

(B) 45°

(C) 55°

(D) 65°

2 When the polynomial $P(x) = x^3 - 5x^2 + kx + 2$ is divided by $(x+1)$ the remainder is 3. What is the value of k? **1**

(A) −7

(B) −5

(C) 5

(D) 7

Marks

3 Which of the following is a simplification of $4\log_e \sqrt{e^x}$? **1**

(A) $4\sqrt{x}$

(B) $\frac{1}{2}x$

(C) $2x$

(D) x^2

4 The acute angle between the lines $2x - y = 0$ and $kx - y = 0$ is equal to $\frac{\pi}{4}$. What is the value of k? **1**

(A) $k = -3$ or $k = -\frac{1}{3}$

(B) $k = -3$ or $k = \frac{1}{3}$

(C) $k = 3$ or $k = -\frac{1}{3}$

(D) $k = 3$ or $k = \frac{1}{3}$

5 Which of the following is a simplification of $\dfrac{1-\cos 2x}{\sin 2x}$? **1**

(A) $1 - \cot 2x$

(B) 1

(C) $\cot x$

(D) $\tan x$

6 The statement $7^n - 3^n$ *is always divisible by* 10 is true for **1**

(A) all integers $n \geq 1$

(B) all integers $n \geq 2$

(C) all odd integers $n \geq 1$

(D) all even integers $n \geq 2$

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Marks

7 What is the value of $\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx$? 1

(A) $\frac{\pi}{6}$
(B) $\frac{\pi}{4}$
(C) $\frac{\pi}{3}$
(D) $\frac{\pi}{2}$

8 The radius r of a circle is increasing at a constant rate of $0\cdot 1\,\text{cm s}^{-1}$. 1
What is the rate at which the area of the circle is increasing when $r = 10\,\text{cm}$?

(A) $\pi\,\text{cm}^2\,\text{s}^{-1}$
(B) $2\pi\,\text{cm}^2\,\text{s}^{-1}$
(C) $10\pi\,\text{cm}^2\,\text{s}^{-1}$
(D) $20\pi\,\text{cm}^2\,\text{s}^{-1}$

9 If $x+\frac{1}{x}=2$ what is the value of $x^2+\frac{1}{x^2}$? 1

(A) 2
(B) 4
(C) 6
(D) 8

10 A particle is performing Simple Harmonic Motion in a straight line. In 1 minute of its motion it completes exactly 15 oscillations and travels exactly 120 metres. What is the amplitude of the motion? 1

(A) 2 metres
(B) 4 metres
(C) 8 metres
(D) 16 metres

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Marks

Section II

60 Marks
Attempt Questions 11-14
Allow about 1 hour and 45 minutes for this section.

Answer the questions in writing booklets provided. Use a separate writing booklet for each question. In Questions 11-14 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet.**

(a) Solve the inequality $\frac{1}{|x-1|} > \frac{1}{2}$. 2

(b) $A(-2,5)$ and $B(6,-7)$ are two points. Find the coordinates of the point $P(x,y)$ that divides the interval AB internally in the ratio $3:1$. 2

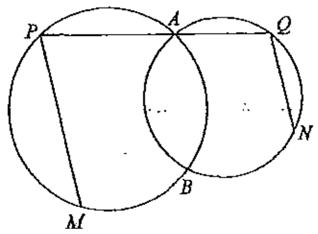
(c) Find $\frac{d}{dx}\left(x^2 \tan^{-1} x\right)$. 2

(d) Use Mathematical Induction to show that for all positive integers $n \geq 1$ 3

$1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \ldots + n\times 2^{n-1} = 1+(n-1)2^n$.

(e) Use the substitution $x = u^2 - 1,\ u \geq 0$, to evaluate $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$. 3

(f) In the diagram the two circles intersect at A and B. PAQ is a straight line and PM is parallel to QN. Copy the diagram. Show that MBN is a straight line. 3



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Marks

Question 12 (15 marks) **Use a separate writing booklet.**

(a) Find the number of ways in which the letters of the word FACTOR can be arranged in a row so that the two vowels are next to each other, but the four consonants are not all next to each other. **2**

(b) The equation $x^3-6x^2+4x+2=0$ has roots α, β and γ. Find the value of

(i) $\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}$. **1**

(ii) $\frac{2}{\alpha}+\frac{2}{\beta}+\frac{2}{\gamma}$. **2**

(c)(i) Given that the limiting sum exists, show that $\tan x+\tan^3 x+\tan^5 x+\ldots=\frac{1}{2}\tan 2x$. **2**

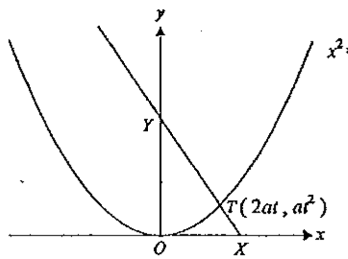
(ii) Hence find the exact value of $\tan\frac{\pi}{8}+\tan^3\frac{\pi}{8}+\tan^5\frac{\pi}{8}+\ldots$. **1**

(d) Solve the equation $\sin^{-1}x=3\cos^{-1}x$, giving the solution correct to 2 decimal places. **3**

(e) In the diagram, $T(2at, at^2)$ is a point on the parabola $x^2=4ay$.

(i) Show that the normal to the parabola at T has equation $x+ty=2at+at^3$. **2**

(ii) This normal cuts the x and y axes at X and Y respectively. Show $\frac{TX}{TY}=\frac{t^2}{2}$. **2**



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Marks

Question 13 (15 marks) **Use a separate writing booklet.**

(a) Find the domain and range of the function $f(x)=\cos^{-1}(2x-1)-\frac{\pi}{2}$. **2**

(b) α is the real root of the equation $\log_e x-\frac{1}{x}=0$. Use one application of Newton's Method with an initial approximation $\alpha_0=1{\cdot}5$ to find the next approximation of α correct to 1 decimal place. **2**

(c) The region bounded by the curve $y=\cos x$ and the x axis between $x=0$ and $x=\frac{\pi}{3}$ is rotated through one complete revolution about the x axis. Find the exact volume of the solid formed. **3**

(d) For each set of tennis that Novak and Andy play there is a probability of $\frac{2}{3}$ that Novak wins the set and a probability of $\frac{1}{3}$ that Andy wins the set.

(i) If Novak and Andy play 4 sets of tennis, find the probability that they win 2 sets each. **2**

(ii) If Novak and Andy continue playing sets of tennis until one of them has won 3 sets, find the probability that Novak wins 3 sets before Andy does. **2**

(e) A particle is moving in a straight line relative to a fixed point O on the line. At time t seconds, it has displacement x metres from O, velocity $v\ \text{ms}^{-1}$ and acceleration $a\ \text{ms}^{-2}$ given by $a=x-2$. Initially the particle is 1 metre to the right of O and is moving away from O at a speed of $1\ \text{ms}^{-1}$.

(i) Show that $v=2-x$. **2**

(ii) Find x in terms of t. **2**

Student name / number ____________________

Marks

Question 14 (15 marks) **Use a separate writing booklet.**

(a) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds it has displacement x metres from a fixed point O on the line given by $x = 6\cos^2 t - 2$.

(i) Show that $\ddot{x} = -4(x-1)$. **2**

(ii) Find the centre and period of the motion. **2**

(b) At time t years the number N of individuals in a population is given by $N = \dfrac{a}{1+be^{-t}}$ for some constants $a>0$ and $b>0$. The initial population size is 20 and the limiting population size is 100.

(i) Show that $\dfrac{dN}{dt} = N\left(1-\dfrac{N}{a}\right)$. **2**

(ii) Find the values of a and b. **2**

(c) A particle is projected from a point O with speed $49\ \text{ms}^{-1}$ at an angle α above the horizontal, where $\alpha > 45°$. It moves in a vertical plane under gravity where the acceleration due to gravity is $9\cdot8\ \text{ms}^{-2}$. At time t seconds its horizontal and vertical displacements from O, x metres and y metres respectively, are given by $x = 49t\cos\alpha$ and $y = 49t\sin\alpha - 4\cdot9t^2$. (Do NOT prove these results).
After 3 seconds the path of the projectile is inclined at 45° to the horizontal.

(i) Show that $\sin\alpha - \cos\alpha = 0\cdot6$. **2**

(ii) Hence find the value of α correct to the nearest degree. **2**

(d)(i) Show that $1+(1+x)+(1+x)^2+\ldots+(1+x)^n = \sum_{r=1}^{n} {}^{n+1}C_r\, x^{r-1}$. **2**

(ii) Hence show that ${}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots + {}^nC_2 = {}^{n+1}C_3$ for all positive integers $n \geq 2$. **1**

Section 1 **Questions 1-10 (1 mark each)**

Question	Answer	Solution	Outcomes
1	C	$\angle ACB = 90°$ (*$\angle$ in a semi-circle is a right angle*) $\therefore \angle CBA = 55°$ (*$\angle$ sum of ΔABC is 180°*) $\therefore \angle MCA = 55°$ (*using alternate segment theorem*)	PE3
2	A	$P(-1)=3 \Rightarrow -1-5-k+2=3 \quad \therefore k=-7$	PE3
3	C	$4\log_e \sqrt{e^x} = 4\log_e e^{\frac{1}{2}x} = 4\times\frac{1}{2}x = 2x$	H3
4	B	$\tan\frac{\pi}{4} = 1 = \left\|\frac{k-2}{1+2k}\right\| \quad \therefore \quad 1+2k=k-2 \text{ or } 1+2k=-(k-2)$; $k=-3$, $3k=1$; $\therefore k=-3$ or $k=\frac{1}{3}$	H5
5	D	$\frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \frac{\sin x}{\cos x} = \tan x$	H5
6	D	$7^{x+2}-3^{x+2} = 9(7^x-3^x)+40\times 7^x$ and $7^1-3^1=4$, $7^2-3^2=40$ .. Since the prime factors of 10 are not factors of 9, and 40 is divisible by 10, by the process of Mathematical Induction, the statement cannot be true for odd positive integers n, but is true for even positive integers n.	HE2
7	C	$\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_1^2 = \sin^{-1}1 - \sin^{-1}\frac{1}{2} = \frac{\pi}{2}-\frac{\pi}{6} = \frac{\pi}{3}$	HE4
8	B	$A=\pi r^2 \quad \therefore \frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi\times 10\times 0\cdot 1 = 2\pi \qquad$ *Ans.* 2π cm^2 s^{-1}	HE5
9	A	$x^2+\frac{1}{x^2} = \left(x+\frac{1}{x}\right)^2 - 2.x.\frac{1}{x} = 2^2-2 = 2$	HE3
10	A	If the amplitude is A metres, then $15\times 4A = 120 \quad \therefore A=2 \quad$ *Ans.* 2 metres	HE3

Section II

Question 11

a. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
• deduces that x lies strictly between -1 and 3	1
• excludes $x=1$	1

Answer

$$\frac{1}{|x-1|} > \frac{1}{2} \quad \therefore \quad |x-1|<2 \text{ and } x\neq 1, \quad -2<x-1<2 \text{ and } x\neq 1 \quad \therefore -1<x<1 \text{ or } 1<x<3$$

Q11 (cont)

b. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• finds the x coordinate of P	1
• finds the y coordinate of P	1

Answer

At P, $x=\frac{3\times 6+1\times(-2)}{3+1}=4$, $\quad y=\frac{3\times(-7)+1\times 5}{3+1}=-4 \quad \therefore P(4,-4)$

c. Outcomes assessed: PE5, HE4

Marking Guidelines

Criteria	Marks
• applies the product rule	1
• derives $\tan^{-1}x$	1

Answer

$$\frac{d}{dx}\left(x^2\tan^{-1}x\right) = 2x\tan^{-1}x + \frac{x^2}{1+x^2}$$

d. Outcomes assessed: HE2

Marking Guidelines

Criteria	Marks
• defines an appropriate sequence of statements and verifies that the first is true	1
• uses the truth of the k^{th} statement to simplify the LHS of the $(k+1)^{th}$ statement	1
• establishes the conditional truth of the $(k+1)^{th}$ statement and completes the induction process	1

Answer

Let $S(n)$, $n=1,2,3,\ldots$ be the sequence of statements defined by

$$S(n): 1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \ldots + n\times 2^{n-1} = 1+(n-1)2^n$$

Consider $S(1)$: $1\times 2^0 = 1$ and $1+(1-1)2^1 = 1 \quad \therefore S(1)$ is true.

If $S(k)$ *is true*: $1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \ldots + k\times 2^{k-1} = 1+(k-1)2^k \quad *$

Consider $S(k+1)$:
$$\begin{aligned} LHS &= \left(1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \ldots + k\times 2^{k-1}\right) + (k+1)\times 2^k \\ &= 1+(k-1)2^k + (k+1)2^k \qquad \textit{if } S(k) \textit{ is true, using } * \\ &= 1+(k-1+k+1)2^k \\ &= 1+2k.2^k \\ &= 1+((k+1)-1)2^{k+1} \\ &= RHS \end{aligned}$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true and then $S(3)$ is true and so on. $\therefore S(n)$ is true for all positive integers $n \geq 1$.

Q11 (cont)

e. Outcomes assessed: HE6

Marking Guidelines

Criteria	Marks
• converts the integral by the process of substitution	1
• writes the primitive function in terms of u	1
• uses u limits to evaluate the definite integral	1

Answer

$x = u^2 - 1, \quad u \ge 0$

$dx = 2u\,du$

$x = 0 \Rightarrow u = 1$

$x = 3 \Rightarrow u = 2$

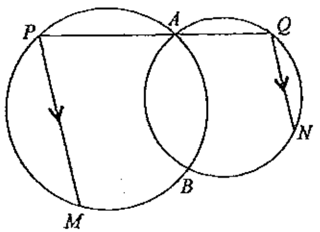
$$\int_0^3 \frac{x}{\sqrt{x+1}}\,dx = \int_1^2 \frac{u^2-1}{u}\cdot 2u\,du$$
$$= 2\left[\tfrac{1}{3}u^3 - u\right]_1^2$$
$$= 2\left\{\tfrac{1}{3}(8-1) - (2-1)\right\}$$
$$= \tfrac{8}{3}$$

f. Outcomes assessed: PE2, PE3

Marking Guidelines

Criteria	Marks
• deduces $\angle MBA$, $\angle ABN$ supplements of $\angle MPA$, $\angle NQA$ respectively	1
• deduces $\angle MPA$, $\angle NQA$ are supplementary	1
• uses these deductions to complete the proof	1

Answer



Construct AB, MB and BN.

$\angle MPA + \angle NQA = 180°$ (*cointerior $\angle$'s within parallel lines are supplementary, $PM \| QN$*)

But $\angle MPA + \angle MBA = 180°$ (*opposite interior $\angle$'s of cyclic quadrilateral ABMP are supplementary*)

$\therefore \angle NQA = \angle MBA$

But $\angle NQA + \angle ABN = 180°$ (*opposite interior $\angle$'s of cyclic quadrilateral ABNQ are supplementary*)

$\therefore \angle MBA + \angle ABN = 180°$

$\therefore \angle MBN = 180°$ (*by addition of adjacent angles*)

Hence MBN is a straight line

Question 12

a. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
• counts the positions of vowel pair and consonants	1
• considers orders of the two vowels and the four consonants	1

Answer

Considering (AO), F, C, T, R, the vowel pair must lie between two consonants (3 possible positions), then the vowels can be ordered in 2 ways and the consonants in 4! ways.
Hence the number of arrangements is $3 \times 2 \times 4! = 144$

b. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
i • writes the sum in terms of the sum of the roots and evaluates using the coefficients	1
ii • expresses the required sum in terms of sums of products of the roots taken 2 and 3 at a time	1
• uses the relationships between roots and coefficients to evaluate the expression	1

Answer

α, β, γ roots of $x^3 - 6x^2 + 4x + 2 = 0$. Hence $\alpha + \beta + \gamma = 6$, $\alpha\beta + \beta\gamma + \gamma\alpha = 4$, $\alpha\beta\gamma = -2$

i. $\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \frac{\alpha+\beta+\gamma}{2} = 3$

ii $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} = \frac{2(\beta\gamma + \gamma\alpha + \alpha\beta)}{\alpha\beta\gamma} = -4$

c. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
i • recognises the sequence as a GP and writes its limiting sum in terms of $\tan x$	1
• uses an appropriate trigonometric identity to establish the result	1
ii • notes that the limiting sum exists and evaluates this sum	1

Answer

i. $\tan x + \tan^3 x + \tan^5 x + \ldots$ is a GP with a limiting sum $\frac{\tan x}{1 - \tan^2 x}$ provided $\tan^2 x < 1$.

Hence for $\tan^2 x < 1$, $\tan x + \tan^3 x + \tan^5 x + \ldots = \frac{1}{2}\left(\frac{2\tan x}{1 - \tan^2 x}\right) = \frac{1}{2}\tan 2x$.

ii. $\tan^2 \frac{\pi}{8} < 1$. Hence $\tan\frac{\pi}{8} + \tan^3\frac{\pi}{8} + \tan^5\frac{\pi}{8} + \ldots = \frac{1}{2}\tan\frac{\pi}{4} = \frac{1}{2}$

Q12 (cont)

d. Outcomes assessed: HE4

Marking Guidelines

Criteria	Marks
• uses an appropriate identity to simplify the equation	1
• finds a trigonometric value for x	1
• evaluates x to the required accuracy	1

Answer

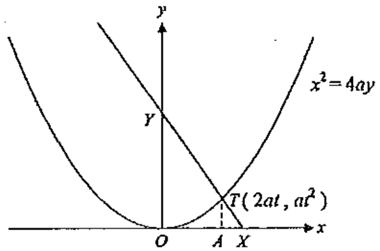
$$\sin^{-1}x = 3\cos^{-1}x \qquad \therefore \cos^{-1}x = \tfrac{\pi}{8}$$
$$\tfrac{\pi}{2} - \cos^{-1}x = 3\cos^{-1}x \qquad x = \cos\tfrac{\pi}{8}$$
$$\tfrac{\pi}{2} = 4\cos^{-1}x \qquad x \approx 0\cdot 92 \quad (\textit{to 2 decimal places})$$

e. Outcomes assessed: PE4, PE5

Marking Guidelines

Criteria	Marks
i • finds the gradient of the normal at T by differentiation	1
• uses this gradient to find the equation of the normal in the required form	1
ii • finds the coordinates of X or Y	1
• finds the ratio in terms of t using TX and TY directly, or projections on either coordinate axis	1

Answer



i.

$$x = 2at \qquad y = at^2$$
$$\frac{dx}{dt} = 2a \qquad \frac{dy}{dt} = 2at$$
$$\therefore \frac{dy}{dx} = \frac{2at}{2a} = t$$

Normal at $T(2at, at^2)$ has gradient $-\frac{1}{t}$ and equation

$$y - at^2 = -\tfrac{1}{t}(x - 2at)$$
$$ty - at^3 = -x + 2at$$
$$x + ty = 2at + at^3$$

ii. Let A be the foot of the perpendicular from T to the x axis. Then $A(2at, 0)$. Also $X(2at + at^3, 0)$. Parallel (vertical) lines OY, AT, together with the vertical line through X, make intercepts in proportion on transversals OX and YX. $\therefore \frac{TX}{TY} = \frac{AX}{AO} = \frac{at^3}{2at} = \frac{t^2}{2}$

Question 13

a. Outcomes assessed: HE4

Marking Guidelines

Criteria	Marks
• finds the domain	1
• finds the range	1

Answer

$f(x) = \cos^{-1}(2x-1) - \frac{\pi}{2}$

Domain

$$-1 \le 2x - 1 \le 1$$
$$0 \le 2x \le 2$$
$$0 \le x \le 1$$

Range

$$0 \le \cos^{-1}(2x-1) \le \pi$$
$$-\tfrac{\pi}{2} \le \cos^{-1}(2x-1) - \tfrac{\pi}{2} \le \pi - \tfrac{\pi}{2}$$
$$-\tfrac{\pi}{2} \le f(x) \le \tfrac{\pi}{2}$$

b. Outcomes assessed: PE3

Marking Guidelines

Criteria	Marks
• differentiates the function	1
• substitutes into the formula for Newton's Method and evaluates	1

Answer

$$f(x) = \log_e x - \frac{1}{x}$$
$$f'(x) = \frac{1}{x} + \frac{1}{x^2} = \frac{x+1}{x^2}$$
$$\frac{f(x)}{f'(x)} = \frac{x^2 \log_e x - x}{x+1}$$

$$\alpha_1 = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)}$$
$$= 1\cdot 5 - \frac{1\cdot 5^2 \log_e 1\cdot 5 - 1\cdot 5}{1\cdot 5 + 1}$$
$$\approx 1\cdot 7$$

c. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• writes a definite integral for the volume in terms of $\cos 2x$	1
• obtains the primitive function and substitutes limits	1
• evaluates the definite integral giving simplified exact answer in surd form in terms of π	1

Answer

$$V = \pi \int_0^{\frac{\pi}{3}} \cos^2 x \, dx$$
$$= \tfrac{\pi}{2} \int_0^{\frac{\pi}{3}} (1 + \cos 2x)\, dx$$
$$= \tfrac{\pi}{2}\left[x + \tfrac{1}{2}\sin 2x\right]_0^{\frac{\pi}{3}}$$
$$= \tfrac{\pi}{2}\left(\tfrac{\pi}{3} + \tfrac{1}{2}\sin\tfrac{2\pi}{3}\right)$$
$$= \tfrac{\pi}{2}\left(\tfrac{\pi}{3} + \tfrac{\sqrt{3}}{4}\right)$$
$$= \tfrac{\pi}{24}\left(4\pi + 3\sqrt{3}\right)$$

Volume is $\frac{\pi}{24}(4\pi + 3\sqrt{3})$ cu. units

Q13 (cont)

d. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • writes an expression for the probability using the binomial distribution	1
• calculates the probability	1
ii • writes an expression for the probability	1
• calculates the probability	1

Answer

i. ${}^4C_2\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^2=\tfrac{8}{27}$

ii. Match could last 3, 4 or 5 sets. Novak must win the final set and exactly 2 of the preceding sets, with probability

$$\left(\tfrac{2}{3}\right)^3+\tfrac{2}{3}\cdot{}^3C_2\left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right)^1+\tfrac{2}{3}\cdot{}^4C_2\left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right)^2=\tfrac{64}{81}$$

e. Outcomes assessed: HE5, HE7

Marking Guidelines

Criteria	Marks
i • finds v^2 as a function of x by integration	1
• interprets the initial and subsequent motion to select the appropriate square root	1
ii • finds t as a function of x by integration	1
• rearranges to get x as a function of t	1

Answer

i. Let the initial direction of travel (to the right relative to O) be the positive direction.

$$\tfrac{1}{2}\frac{dv^2}{dx}=x-2 \qquad v^2=(x-2)^2+c$$

$$\left.\begin{array}{l}t=0\\x=1\\v=1\end{array}\right\}\Rightarrow\quad 1=1+c,\quad \therefore c=0,\quad \therefore v^2=(x-2)^2$$

Initially particle is moving right from $x=1$ while slowing down. If particle were to reach $x=2$, v and a would both be zero and motion could not continue. Hence $1\le x\le 2$ and $v\ge 0$.

$$\therefore v=|x-2|=2-x$$

ii.

$$\frac{dx}{dt}=2-x \qquad \therefore\frac{dt}{dx}=\frac{1}{2-x} \qquad t=-\log_e(2-x)+c$$

$$\left.\begin{array}{l}t=0\\x=1\end{array}\right\}\Rightarrow\quad c=0,\quad e^{-t}=2-x,\quad x=2-e^{-t}$$

Question 14

a. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • finds the velocity as a function of t by differentiation	1
• differentiates a second time to find the acceleration as a function of t then of x	1
ii • states the centre	1
• states the period	1

Answer

i.

$$x=6\cos^2 t-2 = 3(1+\cos 2t)-2 = 1+3\cos 2t$$

$$\dot{x}=-6\sin 2t \qquad \ddot{x}=-12\cos 2t=-4(x-1)$$

ii. The centre of the motion is at $x=1$ and the period is $\frac{2\pi}{2}=\pi$ seconds

b. Outcomes assessed: HE3

Marking Guidelines

Criteria	Marks
i • differentiates N	1
• rearranges into required form	1
ii • finds the value of a	1
• finds the value of b	1

Answer

i.

$$N=\frac{a}{1+be^{-t}}$$

$$\frac{dN}{dt}=\frac{-a\left(-be^{-t}\right)}{\left(1+be^{-t}\right)^2}=N\left(\frac{be^{-t}}{1+be^{-t}}\right)$$

$$\therefore\frac{dN}{dt}=N\left(1-\frac{1}{1+be^{-t}}\right)=N\left(1-\frac{N}{a}\right)$$

ii.

$$\lim_{t\to\infty}N=100\ \Rightarrow\ 100=\frac{a}{1+0}\qquad\therefore a=100$$

$$\left.\begin{array}{l}t=0\\N=20\end{array}\right\}\Rightarrow\quad 20=\frac{a}{1+b},\quad 1+b=\frac{a}{20}\qquad\therefore b=4$$