NSW INDEPENDENT SCHOOLS

2011
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Write your student number and/or name at the top of every page

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ……………………..

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x,\quad x > 0$

Student name / number ....

Question 1 **Begin a new booklet** Marks

(a) Find $\int \frac{x^2+1}{\sqrt{x}}\,dx$. 2

(b) Find $\int \frac{\cos^3 x}{\sin^2 x}\,dx$ using the substitution $u=\sin x$. 3

(c) Evaluate $\int_0^{\frac{1}{2}\log_e 3} \frac{1}{e^x+e^{-x}}\,dx$ using the substitution $u=e^x$. 3

(d) Evaluate in simplest exact form $\int_1^e x^3 \log_e x\;dx$. 3

(e)(i) Using the substitution $t=\tan\frac{x}{2}$, show that 2

$$\int_0^{\frac{\pi}{2}} \frac{1}{5+5\sin x-3\cos x}\,dx=\int_0^1 \frac{1}{4t^2+5t+1}\,dt\ .$$

(ii) Hence evaluate in simplest exact form $\int_0^{\frac{\pi}{2}} \frac{1}{5+5\sin x-3\cos x}\,dx$. 2

Student name / number ....

Question 2 **Begin a new booklet** Marks

(a) If $z_1=2i$ and $z_2=1+3i$, express in the form $a+ib$ (where a and b are real)

(i) $z_1+\overline{z}_2$. 1

(ii) $z_1 z_2$. 1

(iii) $\frac{1}{z_2}$. 1

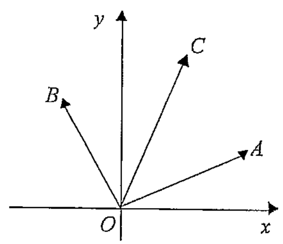
(b)(i) Express $z=1+i\sqrt{3}$ in modulus/argument form. 2

(ii) Hence show that $z^{10}+512z=0$. 2

(c)(i) On an Argand diagram sketch the locus of the point P representing z such that $\left|z-(\sqrt{3}+i)\right|=1$. 2

(ii) Find the set of possible values of $|z|$ and the set of possible principal values of $\arg z$. 2

(d)



In the Argand diagram above, vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ represent the complex numbers z_1, z_2 and z_1+z_2 respectively, where $z_1=\cos\theta+i\sin\theta$ and $z_1+z_2=(1+i)\,z_1$.

(i) Express z_2 in terms of z_1 and show that $OACB$ is a square. 2

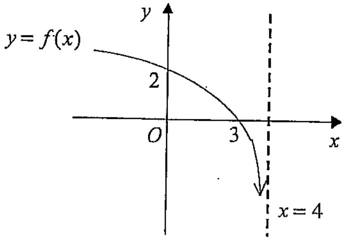
(ii) Show that $(z_1+z_2)\overline{(z_1-z_2)}=2i$. 2

Student name / number ....

Question 3 — Begin a new booklet

Marks

(a) The diagram shows the graph of the curve $y = f(x)$. On separate diagrams, sketch the graphs of the curves listed below, showing clearly intercepts on the coordinate axes and the equations of any asymptotes:



(i) $y = |f(x)|$. — 1

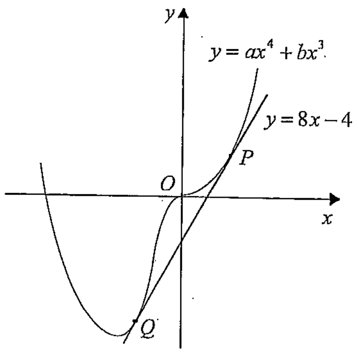
(ii) $y = f(|x|)$. — 1

(iii) $y = f(x^2)$. — 2

(iv) $y = \dfrac{1}{f(x)}$. — 2

(b) $P(x)$ is an even polynomial. Show that when $P(x)$ is divided by $(x^2 - a^2)$, where $a \neq 0$, the remainder is independent of x. — 3

(c)



The line $y = 8x - 4$ is a tangent to the curve $y = ax^4 + bx^3$ (where $a > 0$) at the points P and Q with x coordinates α and β respectively.

(i) Show that the equation $ax^4 + bx^3 - 8x + 4 = 0$ has roots $\alpha, \alpha, \beta, \beta$. — 1

(ii) Show that $\alpha\beta = \dfrac{-2}{\sqrt{a}}$. — 2

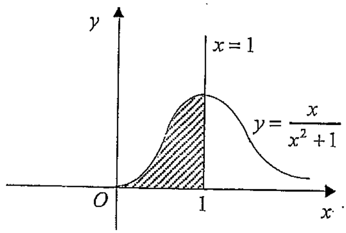
(iii) Hence find the values of a and b. — 3

Student name / number ....

Question 4 — Begin a new booklet

Marks

(a)

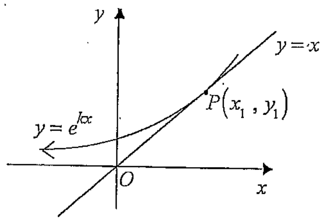


The region bounded by the curve $y = \dfrac{x}{x^2+1}$ and the x-axis between $x = 0$ and $x = 1$ is rotated through one complete revolution about the y-axis.

(i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by $V = 2\pi \displaystyle\int_0^1 \frac{x^2}{x^2+1}\,dx$ — 1

(ii) Hence find the value of V in simplest exact form. — 3

(b)

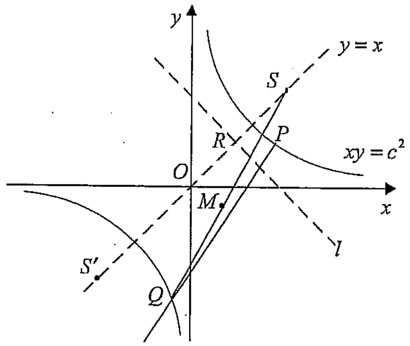


The line $y = x$ is tangent to the curve $y = e^{kx}$ (where $k > 0$) at the point $P(x_1, y_1)$ on the curve. By considering the gradient of OP show that $k = \frac{1}{e}$. — 3

Marks

Question 5 **Begin a new booklet**

(a)



$P\left(cp\,,\frac{c}{p}\right)$ is a variable point on the hyperbola $xy = c^2$ such that $p > 0$.
$S\left(c\sqrt{2}\,,c\sqrt{2}\right)$ is the focus of the hyperbola nearer to P, and the corresponding directrix l has equation $x + y = c\sqrt{2}$. The origin O is the centre of the hyperbola. The directrix l meets OS at R. The normal to the hyperbola at P cuts the hyperbola again at Q. M is the midpoint of QS.

(i) Show that $OS = 2c$ and R is the midpoint of OS. 2

(ii) Show that the normal at P has equation $p^2 x - y = c\left(p^3 - \frac{1}{p}\right)$. 2

(iii) Show that if the parameter at Q is q, then $q\,p^3 = -1$. 1

(iv) Show that as P varies, the coordinates of M satisfy $\left(x - \frac{c}{\sqrt{2}}\right)\left(y - \frac{c}{\sqrt{2}}\right) = \left(\frac{c}{2}\right)^2$. 2

(v) Deduce that the locus of M is one branch of a hyperbola centred at R with foci O and S. 2

(vi) Write down the value of the eccentricity of the hyperbola which contains the locus of M. 1

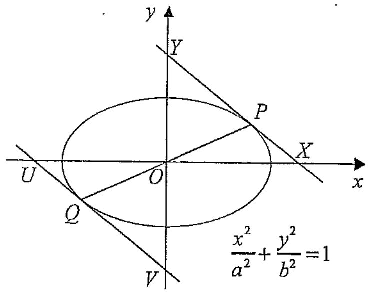
(b) A train of mass 3 tonnes is travelling around a curve of radius 1 km on a track banked so that the outer rail is 3 cm higher than the inner rail, where the rails are 1·5 m apart. The banked track makes an angle θ with the horizontal. When the train has a speed of v ms^{-1}, the normal reaction force exerted by the track on the train is N Newtons and the lateral thrust exerted by one of the rails on the train is F Newtons. Take $g = 10$ ms^{-2}.

(i) By considering the forces acting on the train with $\vec{F}$ directed up the slope, show that $F = 3\cos\theta\left(10\,000\tan\theta - v^2\right)$. 3

(ii) If the lateral thrust F is zero when $v = v_0$, give the magnitude of the lateral thrust when v is 80% of v_0, and state which rail exerts this lateral thrust on the train. 2

Marks

(c)



$P\left(a\cos\theta\,,b\sin\theta\right)$ and $Q\left(a\cos\phi\,,b\sin\phi\right)$ are the endpoints of a diameter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Tangents to the ellipse at P, Q cut the x-axis at X, U respectively, and the y-axis at Y, V respectively. The tangent to the ellipse at P has equation $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$. (Do NOT prove this result).

(i) Show that $\phi = \theta \pm \pi$. 1

(ii) Write down the coordinates of X, Y, U, V in terms of a, b and θ. 2

(iii) Show that the area of $XYUV$ is $\frac{4ab}{|\sin 2\theta|}$. 2

(iv) Find the values of θ, $-\pi < \theta \le \pi$, such the area of $XYUV$ is closest to the area of the ellipse. For one such value of θ, sketch the quadrilateral $XYUV$ and the ellipse, showing carefully the nature of any intersections, the intercepts the quadrilateral makes on the coordinate axes, and the coordinates of P and Q. 3

Student name / number ....

Question 6 **Begin a new booklet** **Marks**

A particle of mass m kg is moving in a medium where the resistance is proportional to the speed. When the particle falls in this medium its terminal velocity is V ms^{-1}. The particle is projected vertically upwards with speed V, reaching a greatest height of H metres above its point of projection. The acceleration due to gravity is g ms^{-2}.

(i) If the resistance to motion has magnitude mkv, $k>0$, by considering forces acting on the particle, show that when it is falling $\ddot{x} = g - kv$. Hence express k in terms of V and g, giving a reason. **2**

(ii) By considering forces acting on the particle, show that when it is moving upwards $\ddot{x} = -\frac{g}{V}(V+v)$. **1**

(iii) By integration, show that $H = \frac{V^2}{g}(1-\ln 2)$. **3**

(iv) Given that $\ddot{x} = \frac{g}{V}(V-v)$ when the particle is falling, show by integration that its speed v on its return to its projection point satisfies $\left(1+\frac{v}{V}\right) + \ln\left\{\frac{1}{2}\left(1-\frac{v}{V}\right)\right\} = 0$. **4**

(v) Let $f(u) = (1+u) + \ln\left\{\frac{1}{2}(1-u)\right\}$. By considering the graphs of $y = -(1+u)$ and $y = \ln\left\{\frac{1}{2}(1-u)\right\}$, show that $f(u)=0$ has a root u such that $0<u<1$. **2**

(vi) Using Newton's formula with a first approximation of $u_0 = 0\cdot 6$, find a second approximation to $f(u)=0$, $0<u<1$. **2**

(vii) What approximate percentage of its terminal velocity has the particle acquired on its return to its point of projection? **1**

Student name / number ....

Question 7 **Begin a new booklet** **Marks**

(a)(i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

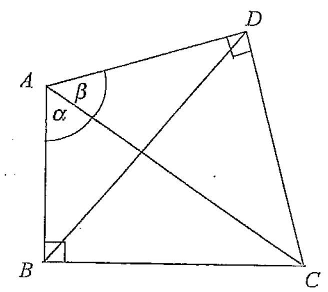
(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}\,dx$. **3**

(b) Let $I_n = \int_0^1 (1-x^r)^n\,dx$, where $r>0$, for $n = 0, 1, 2, \ldots$.

(i) Show that $I_n = \frac{nr}{nr+1} I_{n-1}$ for $n = 1, 2, 3, \ldots$. **3**

(ii) Hence evaluate $\int_0^1 (1-x^{\frac{3}{2}})^3\,dx$. **2**

(c)



$ABCD$ is a quadrilateral in which $\angle ABC = \angle ADC = \frac{\pi}{2}$, $\angle CAB = \alpha$, $\angle CAD = \beta$ and $AC = 1$.

(i) Show that $\angle BDC = \alpha$. **2**

(ii) Hence show that $BD = \sin(\alpha+\beta)$. **3**

Student name / number ....

Question 8 **Begin a new booklet** **Marks**

(a) Let p_n, $n = 1, 2, 3, \ldots$ be the number of ways in which $2n$ distinct items can be paired off (i.e. placed in n groups of 2 where the order of the groups is unimportant).

(i) Clearly $p_1 = 1$. Show that $p_2 = 3$. **1**

(ii) Suppose that A and B are two of $2(k+1)$ distinct items, $k = 2, 3, 4, \ldots$. By counting pairs if A and B are together, and pairs if A and B are separated, show that $p_{k+1} = p_k + 2k(2k-1)\,p_{k-1}$, $k = 2, 3, 4, \ldots$ **2**

(iii) Show by Mathematical Induction that $p_n = \dfrac{(2n)!}{2^n n!}$, $n = 1, 2, 3, \ldots$. **4**

(iv) Adam, Byron, Charles and Donald are four friends. They enter a tennis tournament where the 16 participants are to be assigned randomly to doubles pairings. What is the probability that the four friends comprise two such pairs ? **2**

(b)(i) Show that for $x > 0$, $x > \log_e(1+x)$. **2**

(ii) Hence show that $e^{{}^nC_2} > n!$ for all positive integers $n = 1, 2, 3, \ldots$. **4**

Question 1

a. Outcomes assessed : H8

Marking Guidelines

Criteria	Marks
• rearranges integrand	1
• writes primitive	1

Answer

$$\int \frac{x^2+1}{\sqrt{x}}\ dx = \int \left(x^{\frac{3}{2}} + x^{-\frac{1}{2}}\right)\ dx = \tfrac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + c = \tfrac{2}{5}\sqrt{x}\left(x^2+5\right)+c$$

b. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• converts to an indefinite integral in terms of u	1
• finds the primitive in terms of u	1
• writes primitive as a function of x	1

Answer

$u = \sin x$

$du = \cos x\ dx$

$$\frac{\cos^3 x}{\sin^2 x} = \frac{1-\sin^2 x}{\sin^2 x}\ \cos x$$

$$\int \frac{\cos^3 x}{\sin^2 x}\ dx = \int \frac{1-u^2}{u^2}\ du$$
$$= \int \left(\frac{1}{u^2}-1\right)\ du$$
$$= -\frac{1}{u} - u + c$$
$$= -\left(\operatorname{cosec} x + \sin x\right) + c$$

c. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• converts to definite integral in terms of u	1
• finds the primitive as a function of u	1
• evaluates by substitution of limits	1

Answer

$u = e^x$ $\quad x = 0 \Rightarrow u = 1$

$du = e^x\ dx$ $\quad x = \frac{1}{2}\ln 3 \Rightarrow u = \sqrt{3}$

$\frac{1}{u}du = dx$

$$\int_0^{\frac{1}{2}\log_e 3} \frac{1}{e^x + e^{-x}}\ dx = \int_1^{\sqrt{3}} \frac{1}{u+u^{-1}} \cdot \frac{1}{u}\ du$$
$$= \int_1^{\sqrt{3}} \frac{1}{u^2+1}\ du$$
$$= \left[\tan^{-1} u\right]_1^{\sqrt{3}}$$
$$= \tfrac{\pi}{3} - \tfrac{\pi}{4}$$
$$= \tfrac{\pi}{12}$$

d. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• applies integration by parts	1
• finds the primitive	1
• evaluates	1

Answer

$$\int_1^e x^3 \log_e x\ dx = \tfrac{1}{4}\left[x^4 \log_e x\right]_1^e - \tfrac{1}{4}\int_1^e x^4 . \frac{1}{x}\ dx$$
$$= \tfrac{1}{4}\left(e^4 - 0\right) - \tfrac{1}{16}\left[x^4\right]_1^e$$
$$= \tfrac{1}{16}\left(3e^4+1\right)$$

e. Outcomes assessed : HE6, E8

Marking Guidelines

Criteria	Marks
i • expresses dx in terms of dt and converts x limits to t limits	1
• writes $\sin x$, $\cos x$ in terms of t and simplifies integrand	1
ii • rearranges integrand as a sum of partial fractions	1
• writes primitive and evaluates by substitution of t values.	1

Answer

i.

$t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\ dx$

$\frac{2}{1+t^2}\ dt = dx$

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{4} = 1$

$$5 + 5\sin x - 3\cos x$$
$$= \frac{5(1+t^2)+10t-3(1-t^2)}{1+t^2}$$
$$= \frac{8t^2+10t+2}{1+t^2}$$
$$= \frac{2(4t^2+5t+1)}{1+t^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{5+5\sin x - 3\cos x}\ dx$$
$$= \int_0^1 \frac{1+t^2}{2(4t^2+5t+1)} \cdot \frac{2}{1+t^2}\ dt$$
$$= \int_0^1 \frac{1}{4t^2+5t+1}\ dt$$

ii. $$\frac{1}{4t^2+5t+1} = \frac{1}{(4t+1)(t+1)} = \tfrac{1}{3}\left\{\frac{4}{(4t+1)} - \frac{1}{(t+1)}\right\}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{5+5\sin x - 3\cos x}\ dx = \tfrac{1}{3}\int_0^1 \left\{\frac{4}{4t+1} - \frac{1}{t+1}\right\}\ dt$$
$$= \tfrac{1}{3}\left[\ln\left(\frac{4t+1}{t+1}\right)\right]_0^1$$
$$= \tfrac{1}{3}\left(\ln\tfrac{5}{2} - \ln 1\right)$$
$$= \tfrac{1}{3}\ln\tfrac{5}{2}$$

Question 2

a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • evaluates sum	1
ii • evaluates product	1
iii • evaluates reciprocal	1

Answer

$z_1 = 2i, \quad z_2 = 1 + 3i$

i. $z_1 + \overline{z_2} = 2i + (1 - 3i) = 1 - i$

ii. $z_1 z_2 = 2i(1+3i) = -6 + 2i$

iii. $\dfrac{1}{z_2} = \dfrac{1-3i}{(1+3i)(1-3i)} = \dfrac{1-3i}{1+9} = \frac{1}{10} - \frac{3}{10}i$

b. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • expresses in required form with correct modulus	1
• expresses in required form with correct argument	1
ii • uses de Moivre's theorem to find z^{10}	1
• simplifies expression to obtain required result	1

Answer

i. $z = 1 + i\sqrt{3} = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

ii. $z^{10} + 512\,z = 2^{10}\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right) + 2^{10}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (using de Moivre's theorem)

$= 2^{10}\left\{\cos(2\pi + \frac{4\pi}{3}) + i\sin(2\pi + \frac{4\pi}{3}) + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right\}$

$= 2^{10}\left\{-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3} + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right\}$

$= 0$

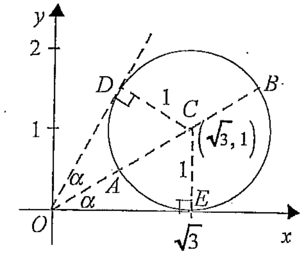
c. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • sketches a circle with correct centre	1
• sketches a circle with correct radius	1
ii • states set of values for $\|z\|$	1
• states set of values for $\arg z$	1

Answer

i. $\left| z - (\sqrt{3} + i) \right| = 1$



ii. $OC = 2$ and $\alpha = \frac{\pi}{6}$

$OA \le |z| \le OB \qquad \therefore 1 \le |z| \le 3$

$0 \le \operatorname{Arg} z \le \angle EOD \qquad \therefore 0 \le \operatorname{Arg} z \le \frac{\pi}{3}$

d. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • expresses z_2 in terms of z_1	1
• explains why $OACB$ is a square	1
ii • uses properties of a square to deduce $z_1 + z_2 = i(z_1 - z_2)$	1
• uses the side and diagonal lengths of the square to complete the proof	1

Answer

i. $z_1 + z_2 = (1+i)\,z_1 \quad \therefore z_2 = i\,z_1$ Hence $\overrightarrow{OB}$ is the rotation of $\overrightarrow{OA}$ anticlockwise by 90°. Hence $OACB$ is a parallelogram in which $OA = OB$ and $\angle AOB = 90°$. $\therefore OACB$ is a square.

ii. The diagonals of a square are equal and meet at right angles.

$\therefore \overrightarrow{OC}$ is the anticlockwise rotation of $\overrightarrow{BA}$ by 90°. Hence

$z_1 + z_2 = i(z_1 - z_2)$

$(z_1 + z_2)\overline{(z_1 - z_2)} = i|z_1 - z_2|^2$

But $BA^2 = OA^2 + OB^2 = 1 + 1 \Rightarrow |z_1 - z_2|^2 = 2$.

$\therefore (z_1 + z_2)\overline{(z_1 - z_2)} = 2i$

Question 3

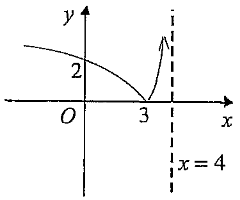
a. Outcomes assessed : E6

Marking Guidelines

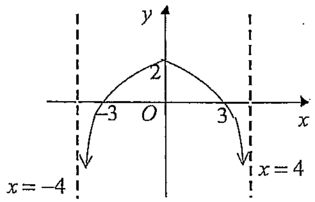
Criteria	Marks
i • copies curve for $x \le 3$ and reflects section of curve for $x > 3$ in x-axis	1
ii • copies curve for $x \ge 0$ and includes reflection of this section of curve in the y-axis	1
iii • sketches curve that is concave down, symmetric in the y-axis, with turning point $(0,2)$	1
• shows asymptotes and x-intercepts	1
iv • shows vertical asymptote $x=3$ and sketches left hand branch correctly	1
• sketches right hand branch correctly showing nature at $x=4$	1

Answer

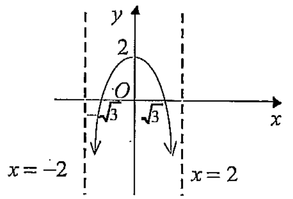
i. $y = |f(x)|$



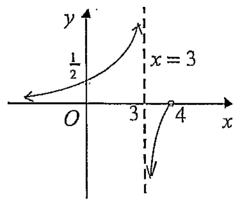
ii. $y = f(|x|)$



iii. $y = f(x^2)$



iv. $y = \dfrac{1}{f(x)}$



b. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
• states remainder on division by (x^2-a^2) is $(cx+d)$ for some constants c and d	1
• uses definition of an even function to deduce $ca+d=-ca+d$	1
• completes proof by showing $c=0$	1

Answer

$P(x) \equiv (x^2-a^2)Q(x)+cx+d$ for constants c, d where $cx+d$ is the remainder on division by x^2-a^2.

$P(x)$ even $\Rightarrow P(-a)=P(a)$ $\quad \therefore ca+d=-ca+d$

$2ca=0$ $\quad$ But $a \ne 0$ $\quad \therefore c=0$.

Hence remainder is some constant d, which is independent of x.

c. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
i • explains why the roots of the given equation are $\alpha,\ \alpha,\ \beta,\ \beta$	1
ii • uses the product of the roots to find $(\alpha\beta)^2$ in terms of a	1
• uses the zero coefficient of x^2 to deduce $\alpha\beta<0$ and then evaluate $\alpha\beta$	1
iii • finds two equations in α, β, a using the coefficients of x^2 and x	1
• writes these equations in terms of a and b	1
• solves for a and b	1

Answer

i. Line $y=8x-4$ intersects curve $y=ax^4+bx^3$ where x-coordinate satisfies $8x-4=ax^4+bx^3$. Hence the x-coordinates of the intersection points of the line and the curve are the real roots of the equation $ax^4+bx^3-8x+4=0$. Since the line is tangent to the curve at P and Q, α and β must each be repeated roots of this equation. Since the equation has degree 4, its roots are $\alpha, \alpha, \beta, \beta$ where α, β are real.

ii. The product of the roots is $\dfrac{4}{a} \Rightarrow (\alpha\beta)^2 = \dfrac{4}{a}$ $\quad \therefore \alpha\beta = \pm\dfrac{2}{\sqrt{a}}$.

The sum of products taken two at a time is 0 $\Rightarrow \alpha^2+4\alpha\beta+\beta^2=0$, and α, β real $\Rightarrow \alpha^2+\beta^2 \ge 0$.

$\therefore \alpha\beta \le 0$ and hence $\alpha\beta = -\dfrac{2}{\sqrt{a}}$.

iii. Using the relationships between roots and coefficients

$$\alpha^2+4\alpha\beta+\beta^2=0 \quad \text{and} \quad 2\alpha\beta^2+2\alpha^2\beta=\frac{8}{a}$$

$$(\alpha+\beta)^2+2\alpha\beta=0 \quad \text{and} \quad 2\alpha\beta(\alpha+\beta)=\frac{8}{a}$$

$$\frac{b^2}{4a^2}-\frac{4}{\sqrt{a}}=0 \quad \text{and} \quad \left(-\frac{4}{\sqrt{a}}\right)\left(\frac{-b}{2a}\right)=\frac{8}{a}$$

$$\therefore b^2=16a\sqrt{a} \quad \text{and} \quad b=4\sqrt{a} \qquad \therefore a=1 \text{ and } b=4$$

Question 4

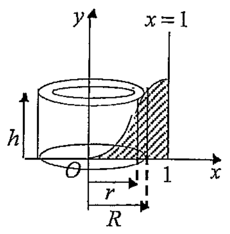
a. Outcomes assessed : H8, E7

Marking Guidelines

Criteria	Marks
i • finds δV in terms of δx then takes limiting sum	1
ii • rearranges integrand	
• finds primitive	1
• evaluates by substitution of limits	1
	1

Answer

i.



$h = \dfrac{x}{x^2+1}$

$r = x$

$R = x + \delta x$

$$\delta V = \pi\left(R^2 - r^2\right)h$$
$$= \pi\left(R+r\right)\left(R-r\right)h$$
$$= \pi(2x+\delta x)(\delta x)\frac{x}{x^2+1}$$

Ignoring terms in $(\delta x)^2$,

$$V = \lim_{\delta x \to 0}\sum_{x=0}^{x=1}\pi\frac{2x^2}{x^2+1}\delta x$$
$$= 2\pi\int_0^1 \frac{x^2}{x^2+1}\,dx$$

ii. $V = 2\pi\displaystyle\int_0^1\left(1-\frac{1}{x^2+1}\right)dx = 2\pi\left[x - \tan^{-1}x\right]_0^1$ $\quad\therefore V = \dfrac{\pi}{2}(4-\pi)$

b. Outcomes assessed : E6

Marking Guidelines

Criteria	Marks
• differentiates to obtain gradient of tangent at P	1
• uses gradient of OP is 1 to deduce $x_1 = y_1 = \frac{1}{k}$	1
• substitutes in equation of curve to find k.	1

Answer

$y = e^{kx}$ $\quad\therefore \dfrac{dy}{dx} = ke^{kx}$. Hence tangent at P has gradient $ke^{kx_1} = ky_1$, since $y_1 = e^{kx_1}$.

But gradient of OP is 1 (since P lies on line $y = x$) $\quad\therefore ky_1 = 1$ and hence $x_1 = y_1 = \dfrac{1}{k}$.

Then since P lies on $y = e^{kx}$, $\dfrac{1}{k} = e^{k\cdot\frac{1}{k}}$ $\quad\therefore k = \dfrac{1}{e}$.

c. Outcomes assessed : E3, E4

Marking Guidelines

Criteria	Marks
i • uses O, P, Q collinear to deduce result	1
ii • writes the coordinates of two of the points	1
• writes the coordinates of the remaining two points	1
iii • deduces that $XYUV$ is a rhombus	1
• expresses the area of the rhombus in terms of its diagonal lengths to obtain the result	1
iv • compares the areas of the quadrilateral and ellipse to deduce that $\lvert\sin 2\theta\rvert = 1$	1
• states the four values of θ	1
• sketches the ellipse inscribed in the quadrilateral giving the required detail.	1

Answer

i. Gradients of OP, OQ are $\dfrac{b}{a}\tan\theta$, $\dfrac{b}{a}\tan\phi$ respectively.

Hence O, P, Q collinear $\Rightarrow \tan\phi = \tan\theta$. Since $\phi \neq \theta$, $\phi = \theta \pm \pi$.

ii. $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ cuts x- and y-axes respectively at $X\left(\dfrac{a}{\cos\theta}, 0\right)$ and $Y\left(0, \dfrac{b}{\sin\theta}\right)$.

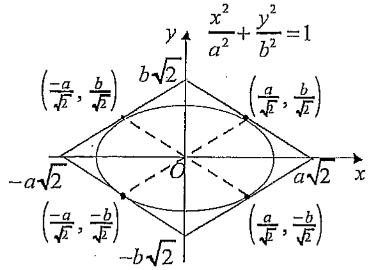
Similarly, since $\cos\phi = -\cos\theta$ and $\sin\phi = -\sin\theta$, $U\left(-\dfrac{a}{\cos\theta}, 0\right)$ and $V\left(0, -\dfrac{b}{\sin\theta}\right)$.

iii. $XYUV$ is a rhombus since the diagonals XU and YV bisect each other at right angles at O.

Hence the area of $XYUV$ is given by $\frac{1}{2}XU.YV = \dfrac{1}{2}\cdot\left|\dfrac{2a}{\cos\theta}\right|\left|\dfrac{2b}{\sin\theta}\right| = \dfrac{4ab}{|\sin 2\theta|}$.

iv. $\dfrac{4ab}{|\sin 2\theta|} \geq 4ab$ with equality iff $\sin 2\theta = \pm 1$. The area of the ellipse is πab, where $\pi < 4$.

Hence the area of $XYUV$ is closest to the area of the ellipse for $\theta = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$.



For any one of the listed angles θ, the quadrilateral makes the same intercepts on the coordinate axes and its four sides are tangents to the ellipse. In each case the diameter PQ is one of the dotted lines, parallel to a pair of quadrilateral sides. The position of P is any one of the four points of contact of the quadrilateral sides with the ellipse, Q being the other end of the corresponding diameter.

Question 5

a. Outcomes assessed : E3, E4

Marking Guidelines

Criteria	Marks
i • shows $OS = 2c$	1
• finds the coordinates of R and deduces R is the midpoint of OS	1
ii • differentiates to find gradient of the tangent in terms of the parameter	1
• deduces gradient of normal at P in terms of p and hence finds the equation of the normal	1
iii • substitutes the coordinates of Q into the equation of the normal at P and simplifies	1
iv • finds the coordinates of M in terms of p	1
• eliminates p to obtain required Cartesian equation	1
v • uses the results from (i) to deduce the centre and foci of the hyperbola containing the locus	1
• explains why M is restricted to lie on the branch nearer to O.	1
vi • states the eccentricity	1

Answer

i. $S\left(c\sqrt{2}, c\sqrt{2}\right)$ $\quad \therefore OS^2 = 2c^2 + 2c^2 = 4c^2 \quad \therefore OS = 2c$

At R, $y = x$ and $x + y = c\sqrt{2}$. $\quad \therefore x = y = \dfrac{c\sqrt{2}}{2}$ $\quad$ Hence R is the midpoint of OS.

ii. $x = ct \Rightarrow \dfrac{dx}{dt} = c \qquad y = \dfrac{c}{t} \Rightarrow \dfrac{dy}{dt} = \dfrac{-c}{t^2}$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt} = -\dfrac{1}{t^2}$

Hence normal at P has gradient p^2 and equation $\quad y - \frac{c}{p} = p^2\left(x - cp\right)$

$$p^2x - y = c\left(p^3 - \tfrac{1}{p}\right)$$

iii. $Q\left(cq, \frac{c}{q}\right)$ lies on both the hyperbola and the normal at P.

$\therefore p^2cq - \frac{c}{q} = c\left(p^3 - \frac{1}{p}\right)$

$$\therefore cp^2(p-q) = c\left(\tfrac{1}{p} - \tfrac{1}{q}\right) = c\frac{-(p-q)}{pq}$$

$$\therefore \quad qp^3 = -1$$

iv. $Q\left(-\frac{c}{p^3}, -cp^3\right) \quad \therefore M\left(\frac{c}{2}(\sqrt{2} - \frac{1}{p^3}), \frac{c}{2}(\sqrt{2} - p^3)\right)$

At M, $\quad x - \dfrac{c}{\sqrt{2}} = \dfrac{-c}{2p^3}$ and $y - \dfrac{c}{\sqrt{2}} = \dfrac{-cp^3}{2}$. $\qquad \therefore \left(x - \dfrac{c}{\sqrt{2}}\right)\left(y - \dfrac{c}{\sqrt{2}}\right) = \left(\dfrac{c}{2}\right)^2$.

v. R has coordinates $\left(\dfrac{c}{\sqrt{2}}, \dfrac{c}{\sqrt{2}}\right)$ and $\quad OR = SR = \frac{1}{2}OS = c = 2\left(\dfrac{c}{2}\right)$.

$\left(x - \dfrac{c}{\sqrt{2}}\right)\left(y - \dfrac{c}{\sqrt{2}}\right) = \left(\dfrac{c}{2}\right)^2$ is the equation of a rectangular hyperbola centred at R with foci lying on $y = x$. But the distance from the centre to either focus is $2\left(\dfrac{c}{2}\right)$ from (i), hence O and S must be the foci of this hyperbola. Considering the coordinates of M in terms of p, since $x < \dfrac{c}{\sqrt{2}}$, $y < \dfrac{c}{\sqrt{2}}$ for $p > 0$, M can only lie on the branch of the hyperbola nearer to O.

vi. The hyperbola containing the locus of M is rectangular hence the eccentricity is $\sqrt{2}$.

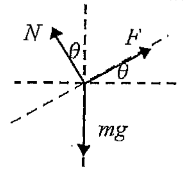
b. Outcomes assessed : E5

Marking Guidelines

Criteria	Marks
i • resolves forces vertically to find one equation in F and N	1
• resolves forces horizontally to find a second equation in F and N	1
• eliminates N and substitutes values to obtain given expression for F	1
ii • finds the magnitude of F	1
• interprets the sign of F to determine which rail exerts the lateral thrust on the train	1

Answer

i. *Forces on the train*



Using Newton's 2nd and 3rd Laws and resolving vertically and horizontally

$$N\cos\theta + F\sin\theta = mg \quad (1)$$

$$N\sin\theta - F\cos\theta = \frac{mv^2}{r} \quad (2)$$

$$\sin\theta \times (1) - \cos\theta \times (2) \Rightarrow F\left(\sin^2\theta + \cos^2\theta\right) = \frac{m\cos\theta}{r}\left(gr\tan\theta - v^2\right)$$

$$F = 3\cos\theta\left(10\,000\tan\theta - v^2\right)$$

ii. For $F = 0$, $v^2 = {v_0}^2 = 10\,000\tan\theta$. When v is 80% of v_0, $v^2 = 0\cdot64 \times 10\,000\tan\theta$

$\therefore F = 3\cos\theta \times 0\cdot36 \times 10\,000\tan\theta$

$= 10\,800\sin\theta$

But $\sin\theta = 0\cdot03 \div 1\cdot5 = 0\cdot02$

$\therefore F = 216$

Since the value of F is positive, $\vec{F}$ is indeed directed up the slope.
Hence the inner rail exerts a lateral thrust of 216 Newtons on the train.

Question 6

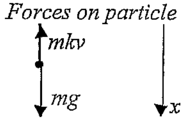
Outcomes assessed : PE3, E5, E6

Marking Guidelines

Criteria	Marks
i • considers forces on particle and uses Newton's 2nd Law to obtain equation of motion	1
• uses limiting value of v as $\ddot{x} \to 0$ to express k in terms of V and g.	1
ii • considers forces on particle and uses Newton's 2nd Law to obtain equation of motion	1
iii • expresses $\dfrac{dx}{dv}$ in terms of v	1
• integrates to find x as a function of v	1
• substitutes $x = H$, $v = 0$ to obtain required expression for H	1
iv • expresses $\dfrac{dx}{dv}$ in terms of v, where x is the distance fallen from the highest point	1
• integrates to find x as a function of v	1
• substitutes $x = H$ to find equation for v on return to point of projection	1
• uses the expression for H in terms of V and g then simplifies equation into required form	1
v • sketches the logarithmic graph showing asymptote and x- and y-intercepts	1
• sketches the line on the same diagram and relates intersection points to roots of $f(u) = 0$	1
vi • obtains $f'(u)$	1
• applies Newton's method to obtain second approximation	1
vii • states approximation percentage	1

Answer

i. *Forces on particle*



By Newton's 2nd Law

$$m\ddot{x} = mg - mkv$$

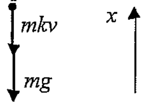
$$\ddot{x} = g - kv$$

Hence $\ddot{x} \to 0$ as $v \to \dfrac{g}{k}$

$\therefore V = \dfrac{g}{k} \qquad \therefore k = \dfrac{g}{V}$

ii. During the upward motion

Forces on particle



Initial conditions

$t=0$

$x=0$

$v=V$

By Newton's 2nd Law,

$m\ddot{x}=-mg-mkv$

$$\ddot{x}=-k\left(\frac{g}{k}+v\right) \qquad \therefore \ddot{x}=-\frac{g}{V}(V+v)$$

iii. $$v\frac{dv}{dx}=-\frac{g}{V}(V+v)$$

$$-\frac{V}{g}\frac{dv}{dx}=\frac{V+v}{v}$$

$$-\frac{g}{V}\frac{dx}{dv}=\frac{v}{V+v}$$

$$-\frac{g}{V}\frac{dx}{dv}=1-V\frac{1}{V+v}$$

$$-\frac{g}{V}x=v-V\ln(V+v)+c$$

$$t=0, x=0, v=V \Rightarrow \quad 0=V-V\ln 2V+c$$

$$\therefore -\frac{g}{V}x=(v-V)-V\ln\left(\frac{V+v}{2V}\right)$$

$$v=0, x=H \Rightarrow \quad -\frac{g}{V}H=-V-V\ln\tfrac{1}{2}$$

$$=-V(1-\ln 2)$$

$$\therefore H=\frac{V^2}{g}(1-\ln 2)$$

iv. Taking initial conditions $t=0, x=0, v=0$ at start of downward motion (ie at greatest height)

$$v\frac{dv}{dx}=\frac{g}{V}(V-v)$$

$$\frac{V}{g}\frac{dv}{dx}=\frac{V-v}{v}$$

$$-\frac{g}{V}\frac{dx}{dv}=\frac{-v}{V-v}$$

$$-\frac{g}{V}\frac{dx}{dv}=1-V\frac{1}{V-v}$$

$$-\frac{g}{V}x=v+V\ln(V-v)+k$$

$$t=0, x=0, v=0 \Rightarrow \quad 0=0+V\ln V+k$$

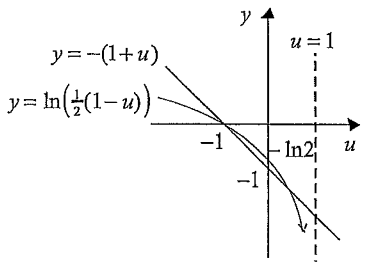
$$\therefore -\frac{g}{V}x=v+V\ln\left(\frac{V-v}{V}\right)$$

$$x=H \Rightarrow -\frac{g}{V}\left(\frac{V^2}{g}(1-\ln 2)\right)=v+V\ln\left(\frac{V-v}{V}\right)$$

Hence the speed on return to projection point is given by $\quad -V+V\ln 2=v+V\ln\left(1-\frac{v}{V}\right)$

$$\left(1+\frac{v}{V}\right)+\ln\left\{\frac{1}{2}\left(1-\frac{v}{V}\right)\right\}=0$$

v.



$f(u)=(1+u)+\ln\left\{\frac{1}{2}(1-u)\right\}$

$f(u)=0$ for u where the line $y=-(1+u)$ intersects the curve $y=\ln\left\{\frac{1}{2}(1-u)\right\}$.

By inspection of the graphs, there is one such intersection point for $0<u<1$.

vi. $$f'(u)=1-\frac{1}{1-u}=\frac{-u}{1-u} \qquad u_1=0\cdot6-\frac{f(0\cdot6)}{f'(0\cdot6)}\approx 0\cdot6-\frac{-0\cdot0094379}{-1\cdot5}\approx 0.59$$

vii. The particle has acquired approximately 60% of its terminal velocity on return to projection point.

Question 7

a. Outcomes assessed : E8

Marking Guidelines

	Criteria	Marks
i	• makes the substitution $u=a-x$	1
	• uses property that value of a definite integral does not depend on the variable of integration	1
ii	• uses the result from (i) to write the given definite integral with $\cos^2 x$ replacing $\sin^2 x$	1
	• uses the table of standard integrals to find the primitive of twice the given integral	1
	• evaluates the given integral by substitution of the limits and rearranging	1

Answer

i. Let $u=a-x$

Then $du=-dx$

and

$x=0 \Rightarrow u=a$

$x=a \Rightarrow u=0$

$$\int_0^a f(x)\,dx=\int_a^0 f(a-u).-du$$

$$=\int_0^a f(a-u)\,du$$

$$=\int_0^a f(a-x)\,dx$$

ii. Let $a=\frac{\pi}{2}$, $f(x)=\dfrac{\sin^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}$. Then $f(\frac{\pi}{2}-x)=\dfrac{\sin^2(\frac{\pi}{2}-x)}{\sqrt{1+(\frac{\pi}{2}-x-\frac{\pi}{4})^2}}=\dfrac{\cos^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}$.

Using (i), if $I=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}\,dx$, then $I=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}\,dx$.

$$2I=\int_0^{\frac{\pi}{2}}\left\{\frac{\sin^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}+\frac{\cos^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}\right\}dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1+(x-\frac{\pi}{4})^2}}\,dx$$

$$=\left[\ln\left\{(x-\tfrac{\pi}{4})+\sqrt{1+(x-\tfrac{\pi}{4})^2}\right\}\right]_0^{\frac{\pi}{2}}$$

$$=\ln\left\{\frac{\frac{\pi}{4}+\sqrt{1+(\frac{\pi}{4})^2}}{-\frac{\pi}{4}+\sqrt{1+(\frac{\pi}{4})^2}}\right\}$$

$$\therefore I=\tfrac{1}{2}\ln\left\{\frac{\pi+\sqrt{16+\pi^2}}{-\pi+\sqrt{16+\pi^2}}\right\}$$

$$=\tfrac{1}{2}\ln\left\{\frac{\left(\pi+\sqrt{16+\pi^2}\right)^2}{\left(16+\pi^2\right)-\pi^2}\right\}$$

$$=\tfrac{1}{2}\ln\left\{\tfrac{1}{4}\left(\pi+\sqrt{16+\pi^2}\right)^2\right\}$$

$$=\ln\left\{\tfrac{1}{4}\left(\pi+\sqrt{16+\pi^2}\right)\right\}$$

b. Outcomes assessed : E8

Marking Guidelines

	Criteria	Marks
i	• applies integration by parts	1
	• evaluates the first part and rearranges the second integrand	1
	• expresses the second integral in terms of I_n, I_{n-1} then rearranges to obtain result	1
ii	• uses the recurrence formula to express I_3 in terms of I_0	1
	• evaluates I_0 and hence evaluates I_3	1

Answer

i. $I_n=\int_0^1 (1-x^r)^n\,dx$, $n=0,1,2,\ldots$ where $r>0$.

For $n=1,2,3,\ldots$

$$I_n=\left[x(1-x^r)^n\right]_0^1-n\int_0^1 x.(1-x^r)^{n-1}.(-rx^{r-1})\,dx$$

$$=0-nr\int_0^1\left\{(1-x^r)-1\right\}(1-x^r)^{n-1}\,dx$$

$$=nr\left\{-I_n+I_{n-1}\right\}$$

$$\therefore (nr+1)I_n=nrI_{n-1}$$

$$I_n=\frac{nr}{nr+1}I_{n-1}$$

ii. For $r=\frac{3}{2}$, $\quad I_3=\dfrac{\left(3\times\frac{3}{2}\right)}{\left(3\times\frac{3}{2}+1\right)}\cdot\dfrac{\left(2\times\frac{3}{2}\right)}{\left(2\times\frac{3}{2}+1\right)}\cdot\dfrac{\left(1\times\frac{3}{2}\right)}{\left(1\times\frac{3}{2}+1\right)}I_0=\dfrac{9}{11}\cdot\dfrac{3}{4}\cdot\dfrac{3}{5}I_0$

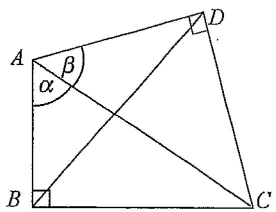
But $I_0=\int_0^1 1\,dx=1$. Hence $I_3=\dfrac{81}{220}$.

c. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
i • explains why ABCD is a cyclic quadrilateral	1
• uses 'angles in the same segment' to deduce result	1
ii • explains why $\sin\angle BCD=\sin(\alpha+\beta)$	1
• explains why $BC=\sin\alpha$	1
• uses the sine rule in ΔBCD to obtain required result	1

Answer



i. $ABCD$ is a cyclic quadrilateral *(opposite angles ABC and ADC are supplementary)*

$\therefore \angle BDC=\angle BAC=\alpha$ *(in circle ABCD, angles subtended at circumference by same arc BC are equal)*

ii. $\angle BCD=\pi-(\alpha+\beta)$ *(opposite angles of a cyclic quadrilateral are supplementary)*

$\therefore \sin\angle BCD=\sin\left\{\pi-(\alpha+\beta)\right\}=\sin(\alpha+\beta)$

Also in ΔABC, $BC=AC\sin\alpha=\sin\alpha$ *(given $AC=1$)*

Hence in ΔBCD, $\dfrac{BD}{\sin\angle BCD}=\dfrac{BC}{\sin\angle BDC}\Rightarrow\dfrac{BD}{\sin(\alpha+\beta)}=\dfrac{\sin\alpha}{\sin\alpha}=1.$ $\quad\therefore BD=\sin(\alpha+\beta)$

Question 8

a. Outcomes assessed : PE3, HE2, HE3

Marking Guidelines

Criteria	Marks
i • counts the pairings of 4 distinct items	1
ii • finds the number of pairings with AB separated in terms of p_{k-1}	1
• adds the number of pairings with AB together	1
iii • defines a sequence of statements $S(n)$, $n=1,2,3,\ldots$ and establishes the truth of the first two	1
• writes an expression for p_{k+1} conditional on the truth of $S(n)$, $n\le k$ for $k\ge 2$	1
• rearranges this expression into the required form	1
• completes the induction process	1
iv • writes the probability in terms of p_k, $k=2,6,8$	1
• evaluates the probability	1

Answer

i. Consider A, B, C, D. possible pairings are AB and CD, AC and BD, AD and BC. $\therefore p_2=3$

ii. If one pair is AB, then the remaining $2k$ items can be paired in p_k ways.

If A and B are separated, then the partners of A and B can be chosen in $2k(2k-1)$ ways, and then the remaining $2(k-1)$ items can be paired in p_{k-1} ways.

Hence $p_{k+1}=p_k+2k(2k-1)\,p_{k-1}$ for $k=2,3,4,\ldots$.

iii. Define the sequence of statements $S(n)$: $p_n=\dfrac{(2n)!}{2^n n!}$, $\quad n=1,2,3,\ldots$

Consider $S(1)$ *and* $S(2)$: $\quad\dfrac{2!}{2^1 1!}=1=p_1$ and $\dfrac{4!}{2^2 2!}=\dfrac{4.3.2.1}{4.2.1}=3=p_2$ $\quad\therefore S(1)$ and $S(2)$ are true.

If $S(n)$ *is true for* $n\le k$: $\quad p_n=\dfrac{(2n)!}{2^n n!}$, $\quad n=1,2,3,\ldots,k$

Consider $S(k+1)$ *where* $k\ge 2$: $\quad p_{k+1}=p_k+2k(2k-1)\,p_{k-1}$

$$=\frac{(2k)!}{2^k k!}+2k(2k-1)\frac{(2k-2)!}{2^{k-1}(k-1)!}\qquad \textit{if } S(n) \textit{ is true for } n\le k$$

$$=\frac{(2k)!}{2^k k!}\left\{1+2k\right\}$$

$$=\frac{(2k)!}{2^k k!}\cdot\frac{(2k+1)(2k+2)}{2(k+1)}$$

$$=\frac{(2k+2)!}{2^{k+1}(k+1)!}$$

Hence if $k\ge 2$ and $S(n)$ is true for $n\le k$, then $S(k+1)$ is true. But $S(1)$ and $S(2)$ are true, hence $S(3)$ is true. Then $S(n)$ is true for $n\le 3$, hence $S(4)$ is true and so on.

Therefore by Mathematical Induction, $S(n)$ is true for all positive integers n, $n=1,2,3\ldots$

iv. The friends can be paired off in p_2 ways with the remaining 12 players paired off in p_6 ways.

Hence the required probability is $\dfrac{p_2\,p_6}{p_8}=3\times\dfrac{12!}{2^6 6!}\times\dfrac{2^8 8!}{16!}=\dfrac{3.2^2.8.7}{16.15.14.13}=\dfrac{1}{65}$

b. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • defines the function $f(x) = x - \log_e(1+x)$ and shows f is increasing for $x > 0$	1
• deduces $f(x) > 0$ for $x > 0$	1
ii • compares the sums of terms x, $x = 1, 2, 3, ..., n-1$ and $\log_e(1+x)$, $x = 1, 2, 3, ..., n-1$	1
• adds the terms of an AP to simplify the first sum, and uses log laws to simplify the second	1
• deduces that ${}^nC_2 > \log_e(n!)$	1
• applies the function e^x to both sides, noting that it is an increasing function	1

Answer

i. Let $f(x) = x - \log_e(1+x)$, $x \ge 0$.

$f'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0$ for $x > 0$ and $f'(x) = 0$ for $x = 0$.

Hence $f(0) = 0$ and $f(x)$ is an increasing function for $x > 0$. $\therefore f(x) > 0$ for $x > 0$.

$\therefore x > \log_e(1+x)$ for $x > 0$.

ii. $1 + 2 + 3 + ... + (n-1) > \log_e(1+1) + \log_e(1+2) + \log_e(1+3) + ... + \log_e\{1 + (n-1)\}$

$$\frac{n(n-1)}{2} > \log_e(2.3.4.....n)$$

$${}^nC_2 > \log_e(n!)$$

$$\therefore e^{{}^nC_2} > n! \qquad \text{(since } e^x \text{ is an increasing function)}$$