NSW INDEPENDENT SCHOOLS

2015
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in Question 11 – 16
- Write your student number and/or name at the top of every page

Total marks – 100

Section I - Pages 2 – 6

10 marks

Attempt Questions 1 - 10

Allow about 15 minutes for this section

Section II - Pages 7 – 12

90 marks

Attempt Questions 11 – 16

Allow about 2 hours 45 minutes for this section

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$

$\int \frac{1}{x}\,dx = \ln x,\ x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x,\ x > 0$

Marks

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

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1 What is the value of $\lim\limits_{h\to 0}\dfrac{\sqrt{h+1}-1}{h}$? 1

(A) 0

(B) $\frac{1}{2}$

(C) 1

(D) 2

2 A, B, C are three consecutive terms in an arithmetic progression. 1

Which of the following is a simplification of $\dfrac{\sin(A+C)}{\sin B}$?

(A) $2\cos B$

(B) $\sin 2B$

(C) $\cot B$

(D) 1

3 What is the number of asymptotes on the graph of the curve $y=\dfrac{x^2}{x^2-1}$? 1

(A) 1

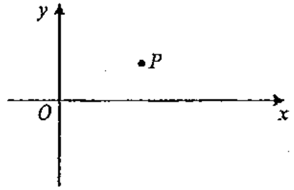
(B) 2

(C) 3

(D) 4

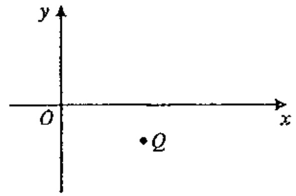
Marks

4 On the Argand diagram below, P represents the complex number z. 1

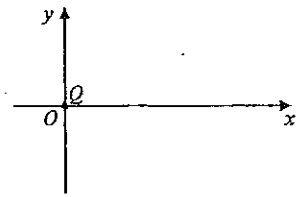


Which of the following Argand diagrams shows the point Q representing $z+\overline{z}$?

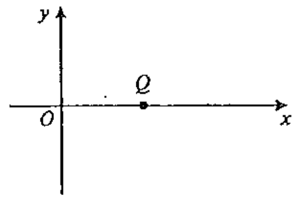
(A)



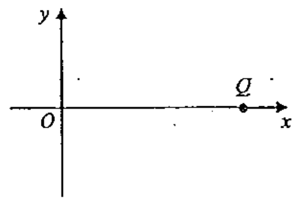
(B)



(C)



(D)



Student name / number ____________________

Marks

5 What is the acute angle between the asymptotes of the hyperbola $\frac{x^2}{3}-y^2=1$? 1

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

6 Which of the following is an expression for $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$ after the substitution $t=\tan\frac{x}{2}$? 1

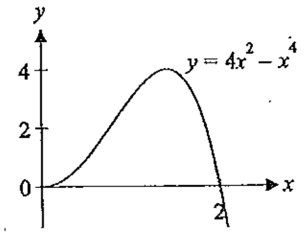
(A) $\int_0^1 \frac{1}{1+2t}\,dt$

(B) $\int_0^1 \frac{2}{1+2t}\,dt$

(C) $\int_0^1 \frac{1}{(1+t)^2}\,dt$

(D) $\int_0^1 \frac{2}{(1+t)^2}\,dt$

7



The region in the first quadrant bounded by the curve $y=4x^2-x^4$ and the x axis between $x=0$ and $x=2$ is rotated through 2π radians about the y axis. Which of the following is an expression for the volume V of the solid formed ? 1

(A) $V=2\pi\int_0^4 \sqrt{4-y}\,dy$

(B) $V=4\pi\int_0^4 \sqrt{4-y}\,dy$

(C) $V=8\pi\int_0^4 \sqrt{4-y}\,dy$

(D) $V=16\pi\int_0^4 \sqrt{4-y}\,dy$

Student name / number ____________________

Marks

8 The equation $x^4+px+q=0$, where $p\neq 0$ and $q\neq 0$, has roots α,β,γ and δ. 1
What is the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$?

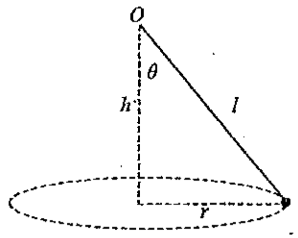
(A) $-4q$

(B) p^2-2q

(C) p^4-2q

(D) p^4

9 1



A particle of mass m attached to a string of length l is suspended from a point O with the string inclined at an angle θ to the vertical where $0<\theta<\frac{\pi}{4}$. The particle moves with constant angular velocity ω in a horizontal circle of radius r at a distance h below O. The forces acting on the particle are the force due to gravity and the tension T in the string. Which of the following statements is NOT correct ?

(A) $T>mg$

(B) $T=ml\omega^2$

(C) $h=\frac{g}{\omega^2}$

(D) $\tan\theta=\frac{g}{r\omega^2}$

10 What is the value of $S=\sum_{r=1}^{\infty} rp(1-p)^{r-1}$ where $0<p<1$? 1

(A) $S=1$

(B) $S=\frac{1}{p}$

(C) $S=\frac{1}{1-p}$

(D) $S=\frac{1}{p(1-p)}$

Student name / number ____________________

Marks

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer the questions in writing booklets provided. Use a separate writing booklet for each question. In Questions 11-16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet.**

(a) If $z=1-2i$ and $w=3+i$ find in the form $a+ib$, where a and b are real, the value of

(i) $z+\overline{w}$. **1**

(ii) $\dfrac{z}{w}$. **2**

(b) The complex number z is given by $z=2(\cos\theta+i\sin\theta)$, where $\frac{\pi}{4}<\theta<\frac{\pi}{2}$.

(i) Express each of z^2 and $\dfrac{1}{z}$ in modulus/argument form. **2**

(ii) On an Argand diagram the points Q and R represent the complex numbers z^2 and $\dfrac{1}{z}$ respectively. If the points Q, O, R are collinear, find z in the form $a+ib$ where a and b are real. **2**

(c) On an Argand diagram shade the region where both $|z-1|\ge 1$ and $-\frac{\pi}{4}\le \arg z\le \frac{\pi}{4}$. **3**

(d) Let $I_n=\int_0^a (a-x)^n \cos x\,dx$, where $a>0$ and $n=0,\ 1,\ 2,\ \ldots$

(i) Show that $I_n=na^{n-1}-n(n-1)I_{n-2}$ for $n\ge 2$. **3**

(ii) Hence find the value of $\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)^2\cos x\,dx$. **2**

Student name / number ____________________

Marks

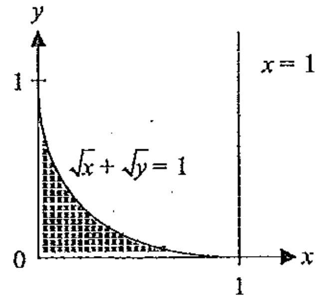
Question 12 (15 marks) **Use a separate writing booklet.**

(a) Find the equation of the tangent to the curve $x^2-xy+y^3=1$ at the point $P(1,1)$ on the curve. **3**

(b) Find $\displaystyle\int \frac{2x-3}{x^2-4x+5}\,dx$. **3**

(c) Use the substitution $x=\tan\theta$ to evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx$. **4**

(d)



The region bounded by the curve $\sqrt{x}+\sqrt{y}=1$ and the x axis between $x=0$ and $x=1$ is rotated through 2π radians about the line $x=1$.

(i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by $V=2\pi\int_0^1(1-x)\left(1-\sqrt{x}\right)^2dx$. **2**

(ii) Hence find the value of V in simplest exact form. **3**

Student name / number ______________________

Question 13 (15 marks) Use a separate writing booklet. **Marks**

(a) The polynomial $P(x)=x^6+ax^3+bx^2$ has a factor $(x+1)^2$. Find the values of the real numbers a and b. **3**

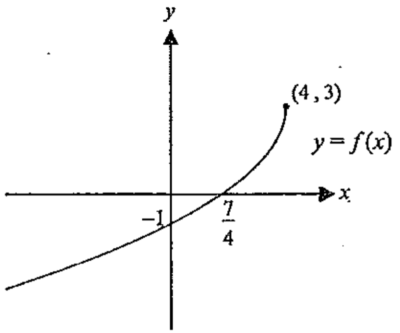
(b) The equation $x^4+bx^3+cx^2+dx+k=0$ has roots α, $\dfrac{1}{\alpha}$, β and $\dfrac{1}{\beta}$.

(i) Show that $k=1$. **1**

(ii) Show that $b=d$. **2**

(c) The equation $x^3+3x^2+2x+1=0$ has roots α, β and γ. Find the monic cubic equation with roots α^2, β^2 and γ^2. **3**

(d)



In the diagram the curve $y=f(x)$ has equation $f(x)=3-2\sqrt{4-x}$.

On separate diagrams sketch the graphs of the following curves showing the coordinates of any endpoints, any intercepts on the axes and the equations of any asymptotes.

(i) $y=\log_e f(x)$. **1**

(ii) $y=f^{-1}(x)$. **1**

(iii) $y=\dfrac{1}{f(x)}$. **2**

(iv) $y=f(x^2)$. **2**

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Question 14 (15 marks) Use a separate writing booklet. **Marks**

(a) A particle of mass m kg is dropped from rest in a medium where the resistance to motion has magnitude $\frac{1}{40}mv^2$ when the speed of the particle is $v\ \mathrm{ms^{-1}}$. After t seconds the particle has fallen x metres. The acceleration due to gravity is $10\ \mathrm{ms^{-2}}$.

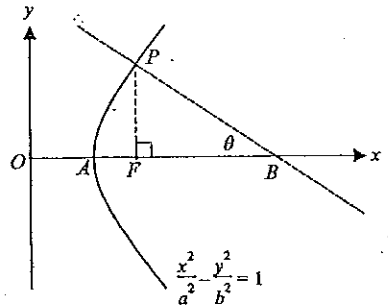
(i) Explain why $\ddot{x}=\frac{1}{40}\left(400-v^2\right)$. **1**

(ii) Find an expression for t in terms of v by integration. **2**

(iii) Show that $v=20\left(1-\dfrac{2}{1+e^t}\right)$. **1**

(iv) Find x as a function of t. **2**

(b)



In the diagram, F is a focus of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ with eccentricity e.

This branch of the hyperbola cuts the x axis at A where $AF=h$. P is the point on the hyperbola vertically above F and the normal at P cuts the x axis at B making an acute angle θ with the x axis.

(i) Show that $\tan\theta=\dfrac{1}{e}$. **3**

(ii) Show that $PF=h(e+1)$. **1**

A bowl is formed by rotating the hyperbola above through one revolution about the x axis. The bowl is then placed on a horizontal table with point A on the table. A particle P of mass m is set in motion around the inside of the bowl, travelling with constant angular velocity ω in a horizontal circle with centre F.

(iii) Show that $\omega^2=\dfrac{g}{he(e+1)}$. **3**

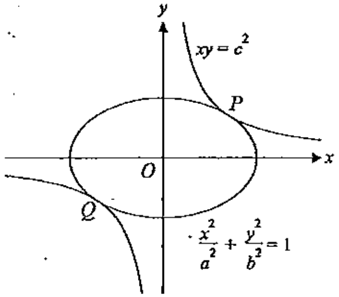
(iv) N is the normal reaction force between the particle P and the bowl. Show that if the hyperbola used to form the bowl is a rectangular hyperbola, then $\dfrac{N}{mg}=\sqrt{\dfrac{3}{2}}$. **2**

Student name / number ______________________

Marks

Question 15 (15 marks) Use a separate writing booklet.

(a) The hyperbola $xy = c^2$, $c > 0$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$ at points P and Q, where $P\left(cp, \frac{c}{p}\right)$ lies in the first quadrant.



(i) Explain why the equation $(bc)^2 t^4 - (ab)^2 t^2 + (ca)^2 = 0$ has roots $p, p, -p, -p$ where $p > 0$. **2**

(ii) Deduce that $p = \frac{a}{c\sqrt{2}}$ and $ab = 2c^2$. **2**

(iii) Show that if S and S' are the foci of the hyperbola $xy = c^2$, then the quadrilateral with vertices P, S, Q and S' has area $2c(a-b)$. **3**

(b)(i) If n is a positive integer and $c > 0$, by considering the stationary values of $f(x) = \frac{1}{x}\left(\frac{x+c}{n+1}\right)^{n+1}$ for $x > 0$, show that $\frac{1}{x}\left(\frac{x+c}{n+1}\right)^{n+1} \geq \left(\frac{c}{n}\right)^n$ for $x > 0$ and state when equality holds. **3**

(ii) Deduce that for $n+1$ positive real numbers x_i, $i = 1, 2, \ldots, n+1$ **1**

$$\left(\frac{1}{n+1}\sum_{i=1}^{n+1} x_i\right)^{n+1} \geq x_{n+1}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^n.$$

(iii) For any set of n positive real numbers x_i, $i = 1, 2, \ldots, n$ where $n \geq 2$, prove by mathematical induction that their arithmetic mean $\frac{1}{n}(x_1 + x_2 + \ldots + x_n)$ is greater than or equal to their geometric mean $(x_1 x_2 \ldots x_n)^{\frac{1}{n}}$. State when equality holds, justifying your statement. **3**

(iv) Deduce $(n!)^{\frac{1}{n}} < \frac{1}{2}(n+1)$. **1**

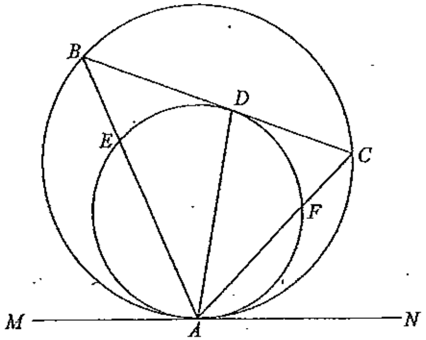
Student name / number ______________________

Marks

Question 16 (15 marks) Use a separate writing booklet.

(a) If $A(x)$ and $B(x)$ are odd polynomial functions show that the product $P(x) = A(x).B(x)$ is an even polynomial function. **2**

(b)



In the diagram, MAN is the common tangent to two circles touching internally at A. B and C are two points on the larger circle such that BC is a tangent to the smaller circle with point of contact D. AB and AC cut the smaller circle at E and F respectively. Copy the diagram. Show that AD bisects $\angle BAC$. **4**

(c) a, b and c are real numbers such that $a > b > c > 1$.

(i) Show that $a^a b^b c^c > a^b b^c c^a$. **2**

(ii) Hence show that $a\ln a + b\ln b + c\ln c > b\ln a + c\ln b + a\ln c$. **2**

(d) In an Argand diagram the points $P, P_1, P_2, P_3, \ldots, P_{n-1}$ represent the roots $1, \omega, \omega^2, \ldots, \omega^{n-1}$ respectively of the equation $z^n - 1 = 0$, where $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

(i) Show that $(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1}) = 1 + z + z^2 + \ldots + z^{n-1}$ and hence show that $PP_1 \times PP_2 \times \ldots \times PP_{n-1} = n$. **2**

(ii) Show that $PP_1^2 + PP_2^2 + \ldots + PP_{n-1}^2 = 2n$. **3**

Section 1 Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1	B	$\lim\limits_{h\to 0}\dfrac{\sqrt{h+1}-1}{h}=\lim\limits_{h\to 0}\dfrac{(h+1)-1}{h\left(\sqrt{h+1}+1\right)}=\lim\limits_{h\to 0}\dfrac{1}{\left(\sqrt{h+1}+1\right)}=\dfrac{1}{2}$	E2
2	A	$B=\frac{1}{2}(A+C)\quad \therefore \dfrac{\sin(A+C)}{\sin B}=\dfrac{\sin 2B}{\sin B}=\dfrac{2\sin B\cos B}{\sin B}=2\cos B$	E2
3	C	$y=\dfrac{x^2}{x^2-1}=1+\dfrac{1}{x^2-1}\quad \therefore x=1,\ x=-1$ and $y=1$ are asymptotes.	E6
4	D	$z+\overline{z}=2\text{Re}z$, so Q lies on the x axis with twice the x coordinate of P.	E3
5	C	Asymptotes have gradients $\pm\frac{1}{\sqrt{3}}$. $\tan\theta=\left\lvert\frac{2\sqrt{3}}{3-1}\right\rvert=\sqrt{3}\quad \therefore\theta=\frac{\pi}{3}$	E3
6	D	$t=\tan\frac{x}{2}$ $dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$ $\dfrac{2}{1+t^2}dt=dx$ $x=0\Rightarrow t=0$ $x=\frac{\pi}{2}\Rightarrow t=1$ $1+\sin x=1+\dfrac{2t}{1+t^2}=\dfrac{(1+t)^2}{1+t^2}$ $\therefore\displaystyle\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin x}dx=\int_0^1\frac{1+t^2}{(1+t)^2}\cdot\frac{2}{1+t^2}dt$ $\displaystyle =\int_0^1\frac{2}{(1+t)^2}\,dt$	E8
7	A	$y=4x^2-x^4$ $x^4-4x^2+4=4-y$ $(x^2-2)^2=4-y$ $(x^2-2)=\pm\sqrt{4-y}$ $x^2=2\pm\sqrt{4-y}$ $\delta V=\pi\left(x_1^{\ 2}-x_2^{\ 2}\right)\delta y$ $=\pi\left\{\left(2+\sqrt{4-y}\right)-\left(2-\sqrt{4-y}\right)\right\}\delta y$ $=2\pi\sqrt{4-y}\ \delta y$ $\displaystyle V=\lim_{\delta y\to 0}\sum_{y=0}^{y=4}2\pi\sqrt{4-y}\ \delta y=2\pi\int_0^4\sqrt{4-y}\ dy$	E7
8	A	$\alpha^4+p\alpha+q=0$ $\beta^4+p\beta+q=0$ $\gamma^4+p\gamma+q=0$ $\delta^4+p\delta+q=0$ $\therefore\left(\alpha^4+\beta^4+\gamma^4+\delta^4\right)+p(\alpha+\beta+\gamma+\delta)+4q=0$ $\left(\alpha^4+\beta^4+\gamma^4+\delta^4\right)+\quad 0\quad +4q=0$ $\therefore\left(\alpha^4+\beta^4+\gamma^4+\delta^4\right)=-4q$	E4
9	D	$T\cos\theta=mg\Rightarrow T>mg$ $T\sin\theta=mr\omega^2=ml\sin\theta\,\omega^2\Rightarrow T=ml\omega^2$ $\tan\theta=\dfrac{mr\omega^2}{mg}\Rightarrow h=\dfrac{r}{\tan\theta}=\dfrac{g}{\omega^2}$ and $\tan\theta=\dfrac{r\omega^2}{g}$	E5
10	B	$S=p+2p(1-p)+3p(1-p)^2+4p(1-p)^3+\ldots\quad(1)$ $(1-p)S=\quad p(1-p)+2p(1-p)^2+3p(1-p)^3+\ldots\quad(2)$ $(1)-(2)\Rightarrow\quad pS=p+p(1-p)+p(1-p)^2+p(1-p)^3+\ldots$ $\therefore pS=\dfrac{p}{1-(1-p)}=1\quad\therefore S=\dfrac{1}{p}$	E2

Section II

Question 11

a. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • writes correct sum	1
ii • uses the complex conjugate of w to write with real denominator	1
• expands and simplifies numerator then expresses in required form	1

Answer

i. $z+\overline{w}=1-2i+3-i=4-3i$

ii. $\dfrac{z}{w}=\dfrac{z\overline{w}}{w\overline{w}}=\dfrac{(1-2i)(3-i)}{3^2+1^2}=\dfrac{1-7i}{10}=\frac{1}{10}-\frac{7}{10}i$

b. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • writes expression for square of z	1
• writes expression for reciprocal of z	1
ii • compares arguments to find θ	1
• writes z in required form	1

Answer

i. $z=2(\cos\theta+i\sin\theta)\quad\therefore z^2=4(\cos 2\theta+i\sin 2\theta)$ and $\dfrac{1}{z}=\frac{1}{2}(\cos(-\theta)+i\sin(-\theta))$

ii. $\frac{\pi}{2}<\arg z^2<\pi$ and $-\frac{\pi}{2}<\arg\dfrac{1}{z}<-\frac{\pi}{4}$. $\therefore Q, O, R$ collinear $\Rightarrow\arg z^2=\pi+\arg\dfrac{1}{z}$

$2\theta=\pi+(-\theta)$

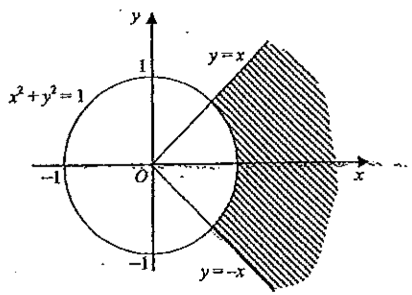
Hence $\theta=\frac{\pi}{3}$ and $z=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=1+\sqrt{3}\,i$

c. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
• shows part of circle, centre origin and radius 1, as a boundary of the region	1
• shows lines $y=\pm x$ as boundaries of the region	1
• shades the correct region	1

Answer



Q 11 (cont)

d. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
i • uses integration by parts, evaluating the first term	1
• uses integration by parts a second time	1
• simplifies to obtain required reduction formula	1
ii • uses the reduction formula	1
• evaluates the remaining definite integral	1

Answer

i.

$$\int_0^a (a-x)^n \cos x\,dx = \left[(a-x)^n \sin x\right]_0^a - \int_0^a (-n)(a-x)^{n-1}\sin x\,dx$$

$$= 0 + n\left[(a-x)^{n-1}(-\cos x)\right]_0^a - n(n-1)\int_0^a (a-x)^{n-2}\cos x\,dx$$

$$\therefore I_n = na^{n-1} - n(n-1)I_{n-2}$$

ii. $a=\frac{\pi}{2}$ and $I_0=\int_0^{\frac{\pi}{2}}\cos x\,dx=\left[\sin x\right]_0^{\frac{\pi}{2}}=1$. $\quad\therefore \int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)^2\cos x\,dx = I_2 = 2\left(\frac{\pi}{2}\right)-2I_0=\pi-2$

Question 12

a. Outcomes assessed: E6

Marking Guidelines

Criteria	Marks
• differentiates implicitly to express derivative in terms of x and y	1
• evaluates this derivative at P to find the gradient of the tangent	1
• finds the equation of the tangent	1

Answer

$$x^2 - xy + y^3 = 1$$

$$2x - \left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$\left(3y^2 - x\right)\frac{dy}{dx} = y - 2x$$

At $P(1,1)$, $\frac{dy}{dx}=\frac{y-2x}{3y^2-x}=-\frac{1}{2}$.

$\therefore$ Tangent at P has gradient $-\frac{1}{2}$ and equation $y-1=-\frac{1}{2}(x-1)$

$$x+2y-3=0$$

b. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• rearranges the integrand into a sum of standard forms	1
• finds one function part of the primitive	1
• finds the second function part of the primitive	1

Answer

$$\int \frac{2x-3}{x^2-4x+5}\,dx = \int \left\{\frac{2(x-2)}{(x-2)^2+1} + \frac{1}{(x-2)^2+1}\right\}dx = \ln\left\{(x-2)^2+1\right\} + \tan^{-1}(x-2) + C$$

Q12 (cont)

c. Outcomes assessed: E8

Marking Guidelines

Criteria	Marks
• transforms integrand	1
• converts limits	1
• writes primitive function	1
• evaluates using limits	1

Answer

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$x=\sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

$x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$x^2\sqrt{1+x^2} = \tan\theta\sin\theta\sec^2\theta$

$$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec}\theta\cot\theta\,d\theta$$

$$= -\left[\operatorname{cosec}\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= -\frac{2}{\sqrt{3}} + \sqrt{2}$$

$$= \frac{1}{3}\left(3\sqrt{2} - 2\sqrt{3}\right)$$

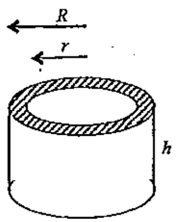
d. Outcomes assessed: E7, E8

Marking Guidelines

Criteria	Marks
i • writes an expression for the volume of a typical cylindrical shell	1
• takes the limiting sum of shell volumes to express V as a definite integral	1
ii • expands the integrand	1
• finds the primitive function	1
• evaluates V in simplest exact form	1

Answer

i.



$R = 1 - x$

$r = 1 - x - \delta x$

$h = \left(1 - \sqrt{x}\right)^2$

$\delta V = \pi(R+r)(R-r)h$

$= \pi\left\{2(1-x) - \delta x\right\}.\delta x.\left(1-\sqrt{x}\right)^2$

Ignoring second order terms,

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{x=1} 2\pi(1-x)\left(1-\sqrt{x}\right)^2\delta x$$

$$= 2\pi\int_0^1 (1-x)\left(1-\sqrt{x}\right)^2 dx$$

ii.

$$V = 2\pi\int_0^1 \left(1 - x^2 - 2\sqrt{x} + 2x\sqrt{x}\right)dx$$

$$= 2\pi\left[x - \frac{1}{3}x^3 - \frac{4}{3}x^{\frac{3}{2}} + \frac{4}{5}x^{\frac{5}{2}}\right]_0^1$$

$$= \frac{4}{15}\pi$$

Question 13

a. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
• uses $P(-1)=0$ to obtain one equation in a and b	1
• uses $P'(-1)=0$ to obtain a second equation in a and b	1
• solves for a and b	1

Answer

$P(x)=x^6+ax^3+bx^2$ $\quad$ $P(-1)=0 \Rightarrow a-b=1$ (1) $\quad$ $(2)-2\times(1) \Rightarrow a=4$

$P'(x)=6x^5+3ax^2+2bx$ $\quad$ $P'(-1)=0 \Rightarrow 3a-2b=6$ (2) $\quad$ $b=3$

b. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
i • uses the product of the roots to deduce the value of k	1
ii • writes a second monic quartic equation satisfied by the reciprocals of the roots	1
• compares coefficients to deduce the result	1

Answer

$x^4+bx^3+cx^2+dx+k=0$ $\quad$ i. $k=\alpha.\frac{1}{\alpha}.\beta.\frac{1}{\beta}=1$

ii. The reciprocals of the roots of $x^4+bx^3+cx^2+dx+1=0$ satisfy $\left(\frac{1}{x}\right)^4+b\left(\frac{1}{x}\right)^3+c\left(\frac{1}{x}\right)^2+d\left(\frac{1}{x}\right)+1=0$

Hence α, $\frac{1}{\alpha}$, β and $\frac{1}{\beta}$ are also the roots of $x^4+dx^3+cx^2+bx+1=0$. This is only possible if the two monic quartic equations are identical. Hence comparing coefficients of x^3 or x , $b=d$.

c. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
• writes an equation in terms of $\sqrt{x}$ that is satisfied by the squares of α, β, γ	1
• rearranges this equation to eliminate surd expressions	1
• completes the rearrangement to obtain the equation as a monic cubic equation in the usual form	1

Answer

α, β, γ satisfy $x^3+3x^2+2x+1=0$. Hence α^2, β^2, γ^2 satisfy $\left(x^{\frac{1}{2}}\right)^3+3\left(x^{\frac{1}{2}}\right)^2+2\left(x^{\frac{1}{2}}\right)+1=0$.

This second equation can be rearranged to give $x^{\frac{1}{2}}(x+2)=-(1+3x)$ and hence $x(x+2)^2=(1+3x)^2$.

Hence α^2, β^2, γ^2 are roots of $x^3+4x^2+4x=1+6x+9x^2$.

Since the monic cubic equation with roots α^2, β^2, γ^2 is unique, required equation is $x^3-5x^2-2x-1=0$.

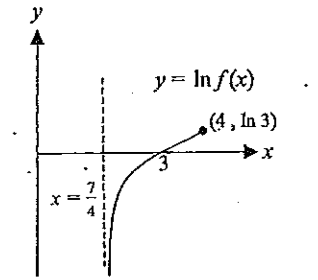
d. Outcomes assessed: E6

Marking Guidelines

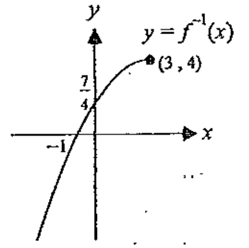
Criteria	Marks
i • correct shape with endpoint, x intercept and asymptote shown	1
ii • correct shape with x and y intercepts and endpoint shown	1
iii • left branch correct shape with asymptote and y intercept shown	1
• second branch correct shape with endpoint shown	1
iv • correct shape and position with y intercept shown	1
• x intercepts and endpoints shown	1

Answer

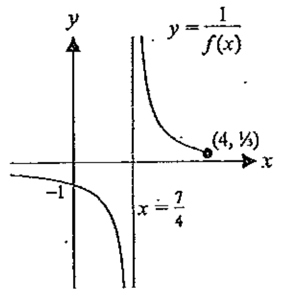
i.



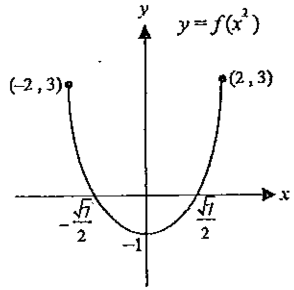
ii



iii.



iv.



Question 14

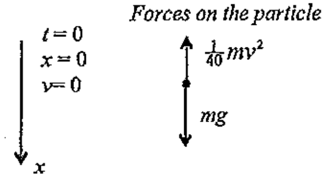
a. Outcomes assessed: E5

Marking Guidelines

Criteria	Marks
i • considers resultant force on the particle and invokes Newton's second law	1
ii • expresses derivative of t with respect to v as sum of partial fractions	1
• finds t as a function of v by integration, using initial conditions to evaluate the constant	1
iii • rearranges to find v as a function of t in required form	1
iv • rearranges derivative of x with respect to t into a standard form for integration	1
• finds x as a function of t by integration, using initial conditions to evaluate the constant	1

Answer

i.



By Newton's second law :

$$m\ddot{x} = mg - \tfrac{1}{40}mv^2$$

$$\ddot{x} = \tfrac{1}{40}\left(400 - v^2\right)$$

ii. $\dfrac{dv}{dt} = \tfrac{1}{40}\left(400 - v^2\right)$

$$\frac{dt}{dv} = \frac{40}{20^2 - v^2}$$

$$= \frac{1}{20+v} + \frac{1}{20-v}$$

$$t = \ln\left(\frac{20+v}{20-v}\right) + c$$

$v = 0$ when $t = 0 \Rightarrow c = 0$

$$\therefore t = \ln\left(\frac{20+v}{20-v}\right)$$

iii. $e^t = \dfrac{20+v}{20-v}$

$$1 + e^t = \frac{40}{20-v}$$

$$\frac{40}{1+e^t} = 20 - v$$

$$v = 20\left(1 - \frac{2}{1+e^t}\right)$$

iv. $\dfrac{dx}{dt} = 20\left(1 - \dfrac{2e^{-t}}{e^{-t}+1}\right)$

$$x = 20\left\{t + 2\ln\left(1 + e^{-t}\right) + c\right\}$$

$$\left.\begin{array}{l} x = 0 \\ t = 0 \end{array}\right\} \Rightarrow c = -2\ln 2$$

$$\therefore x = 20\left\{t + 2\ln\left(\frac{1+e^{-t}}{2}\right)\right\}$$

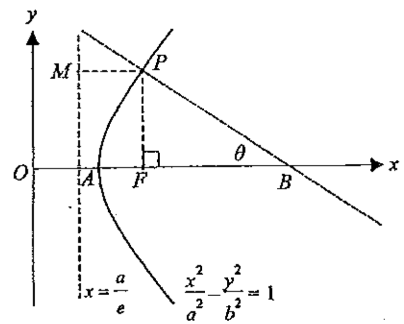
Q14 (cont)

b. Outcomes assessed: E3, E5

Marking Guidelines

Criteria	Marks
i • finds the coordinates of P in terms of a and e	1
• uses differentiation to find the gradient of the normal at P	1
• deduces the value of $\tan\theta$ in terms of e	1
ii • writes h in terms of a and e to deduce required expression for PF	1
iii • resolves forces on P vertically and horizontally, applying Newton's 2nd Law	1
• finds an expression for $\tan\theta$ from the simultaneous equations	1
• uses values of $\tan\theta$ and PF from previous parts to obtain required result	1
iv • finds $\tan\theta$ when the hyperbola is rectangular	1
• uses an appropriate trigonometric identity to deduce the required expression for N.	1

Answer



i. $F(ae, 0)$, $A(a, 0)$. $\quad b^2 = a^2\left(e^2 - 1\right)$.

$$PF = e.PM = e\left(ae - \frac{a}{e}\right) \quad \therefore P\left(ae,\ a(e^2-1)\right)$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

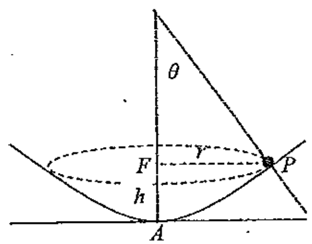
At P, $\quad \dfrac{dy}{dx} = \dfrac{b^2}{a^2}\cdot\dfrac{x}{y} = \left(e^2-1\right)\dfrac{ae}{a\left(e^2-1\right)} = e$

Hence normal to hyperbola at P has gradient

$$-\frac{1}{e} = \tan\left(180° - \theta\right) \quad \therefore \tan\theta = \frac{1}{e}$$

ii. $h = AF = a(e-1)$. $\quad \therefore PF = a\left(e^2-1\right) = h(e+1)$.

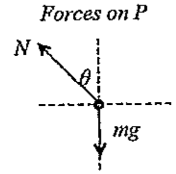
iii.



From i. and ii. :

$$r = h(e+1)$$

$$\tan\theta = \frac{1}{e}$$



Applying Newton's 2nd Law and resolving forces vertically and horizontally :

$$\left.\begin{array}{l} N\cos\theta = mg. \\ N\sin\theta = mr\omega^2 \end{array}\right\} \therefore \quad \begin{array}{l} \tan\theta = \dfrac{r\omega^2}{g}. \\ \dfrac{1}{e} = \dfrac{h(e+1)\omega^2}{g} \end{array} \quad \therefore \omega^2 = \frac{g}{he(e+1)}$$

iv. $e = \sqrt{2} \Rightarrow \tan\theta = \frac{1}{\sqrt{2}} \quad \therefore \sec\theta = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}} \quad \therefore \dfrac{N}{mg} = \dfrac{1}{\cos\theta} = \sqrt{\dfrac{3}{2}}$

Question 15

a. Outcomes assessed: E3, E4

Marking Guidelines

Criteria	Marks
i • explains why this equation gives the hyperbola parameter value at any intersection point	1
• deduces the nature of the roots from the fact that the curves touch at P	1
ii • relates sum of products of roots taken two at a time and coefficients to obtain expression for p	1
• relates product of the roots and coefficients to express ab in terms of c	1
iii • deduces quadrilateral is a parallelogram and finds length of diagonal SS'	1
• finds perpendicular distance from P to SS' in terms of a, b, c	1
• finds expression for area of quadrilateral and rearranges into required form	1

Answer

i. The hyperbola intersects the ellipse at a point with coordinates $\left(ct,\frac{c}{t}\right)$ when $\frac{c^2t^2}{a^2}+\frac{c^2}{t^2b^2}=1$.

Rearranging this equation gives $(bc)^2t^4-(ab)^2t^2+(ca)^2=0$.
Since the hyperbola touches the ellipse at P, p must be a double root of this equation. Since P lies in the first quadrant, $p>0$. Since the equation is a quadratic in t^2, $-p$ must also be a double root.

ii. $p^2+4p(-p)+(-p)^2=-\frac{(ab)^2}{(bc)^2}$ $\qquad$ $p^2(-p)^2=\frac{(ca)^2}{(bc)^2}$

$\therefore -2p^2=-\left(\frac{a}{c}\right)^2$ $\qquad$ $\therefore\ p^4=\left(\frac{a}{b}\right)^2$

$\therefore p>0 \Rightarrow p=\frac{a}{c\sqrt{2}}$ $\qquad$ $\therefore\left(\frac{a^2}{2c^2}\right)^2=\left(\frac{a}{b}\right)^2$ $\qquad$ $\therefore ab=2c^2$

iii. S, S' lie on the line $y=x$ and O is the midpoint of SS' . Since parameter at Q is $-p$, O is also the midpoint of PQ. Hence the quadrilateral with vertices P, S, Q, S' is a parallelogram (diagonals bisect each other) and the diagonal SS' divides the quadrilateral into two congruent triangles.

$\therefore$ *Area* $PSQS'=2\times$ *Area* $\Delta PSS'$.

Perpendicular distance from $P\left(cp,\frac{c}{p}\right)$ to line $y=x$ is $d=\frac{\left|cp-\frac{c}{p}\right|}{\sqrt{1^2+1^2}}=\frac{c}{\sqrt{2}}\left|p-\frac{1}{p}\right|$.

$p-\frac{1}{p}=\frac{a}{c\sqrt{2}}-\frac{c\sqrt{2}}{a}=\frac{a^2-2c^2}{ac\sqrt{2}}$ $\qquad$ But $a^2-2c^2=a^2-ab$ and $a>b>0$ $\qquad$ $\therefore d=\frac{c}{\sqrt{2}}\cdot\left|\frac{a(a-b)}{ac\sqrt{2}}\right|=\frac{a-b}{2}$

$S\left(c\sqrt{2},c\sqrt{2}\right)$ $\quad\therefore SS'=2\times SO=2\sqrt{2c^2+2c^2}=4c$

$\therefore$ *Area* $PSQS'=d\times SS'=2c(a-b)$.

Q15 (cont)

b. Outcomes assessed: HE2, E2

Marking Guidelines

Criteria	Marks
i • derives $f(x)$ with respect to x	1
• finds the value of $x>0$ for which f takes its minimum value	1
• finds this minimum value of f and states when it is attained	1
ii • selects appropriate values of x and c to deduce result	1
iii • defines an appropriate sequence of statements and verifies the first is true	1
• shows that if the k^{th} statement is true, then the $(k+1)^{th}$ is conditionally true	1
• completes the induction process and states with explanation when equality holds	1
iv • selects an appropriate set of positive real numbers to deduce the result	1

Answer

i. $f(x)=\frac{1}{x}\left(\frac{x+c}{n+1}\right)^{n+1}$

$f'(x)=\frac{-1}{x^2}\left(\frac{x+c}{n+1}\right)^{n+1}+\frac{1}{x}\left(\frac{x+c}{n+1}\right)^{n}$
$=\frac{1}{x^2}\left(\frac{x+c}{n+1}\right)^{n}\left(x-\frac{x+c}{n+1}\right)$
$=\frac{1}{x^2}\left(\frac{x+c}{n+1}\right)^{n}\left(\frac{nx-c}{n+1}\right)$

For $x>0$, only stationary value is for $nx=c$.

$0<x<\frac{c}{n}\Rightarrow f'(x)<0$ and $x>\frac{c}{n}\Rightarrow f'(x)>0$

$\therefore f(x)$ has minimum stationary value at $x=\frac{c}{n}$

$\therefore f(x)\geq f\left(\frac{c}{n}\right)=\frac{n}{c}\left(\frac{c}{n}\right)^{n+1}=\left(\frac{c}{n}\right)^n$ for $x>0$

with equality if and only if $x=\frac{c}{n}$.

ii. If $c=\sum_{i=1}^{n}x_i$ and $x=x_{n+1}$, then $x_{n+1}+c=\sum_{i=1}^{n+1}x_i$ and $\frac{1}{x_{n+1}}\left(\frac{1}{n+1}\sum_{i=1}^{n+1}x_i\right)^{n+1}\geq\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^n$.

Hence $\left(\frac{1}{n+1}\sum_{i=1}^{n+1}x_i\right)^{n+1}\geq x_{n+1}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^n$ (with equality if and only if $x_{n+1}=\frac{1}{n}\sum_{i=1}^{n}x_i$).

iii. Let $S(n)$, $n=2,3,\ldots$ be the sequence of statements defined by

$S(n)$: $\frac{1}{n}(x_1+x_2+\ldots+x_n)\geq(x_1x_2\ldots x_n)^{\frac{1}{n}}$ *for any set of* n *positive real numbers* x_i, $i=1,2,\ldots,n$.

Consider $S(2)$: $\left\{\frac{1}{2}(x_1+x_2)\right\}^2\geq x_2(x_1)^1$ from (ii), hence $\frac{1}{2}(x_1+x_2)\geq(x_1x_2)^{\frac{1}{2}}$ and $S(2)$ is true.

If $S(k)$ *is true* : $\frac{1}{k}(x_1+x_2+\ldots+x_k)\geq(x_1x_2\ldots x_k)^{\frac{1}{k}}$ *

Consider $S(k+1)$: $\left\{\frac{1}{k+1}(x_1+x_2+\ldots+x_{k+1})\right\}^{k+1}\geq x_{k+1}\left\{\frac{1}{k}(x_1+x_2+\ldots+x_k)\right\}^k$ from (ii)
$\geq x_{k+1}(x_1x_2\ldots x_k)$ *if* $S(k)$ *is true using* *

giving $\frac{1}{k+1}(x_1+x_2+\ldots+x_{k+1})\geq(x_1x_2\ldots x_{k+1})^{\frac{1}{k+1}}$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(2)$ is true, then $S(3)$ is true and so on. Hence by mathematical induction, $S(n)$ is true for positive integers n, $n\geq 2$.
The arithmetic mean is equal to the geometric mean if and only if $x_1=x_2=\ldots=x_n$, since result in (ii) requires that $x_2=x_1$, $x_3=\frac{1}{2}(x_1+x_2)$, $x_4=\frac{1}{3}(x_1+x_2+x_3)$, ... in induction process for equality to hold.

iv. Considering the numbers 1, 2, 3, ..., n : $\frac{1}{n}(1+2+3+\ldots+n)>(n!)^{\frac{1}{n}}$ $\qquad\therefore (n!)^{\frac{1}{n}}<\frac{1}{2}(n+1)$.

Question 16

a. Outcomes assessed: E4

Marking Guidelines

Criteria	Marks
• writes $P(-x)$ in terms of $A(x)$ and $B(x)$ using the definition of an odd function	1
• simplifies to successfully apply the test for an even function	1

Answer

$A(-x)=-A(x)$ and $B(-x)=-B(x)$

$\therefore P(-x)=A(-x).B(-x)=\{-A(x)\}\{-B(x)\}=A(x).B(x)=P(x)$

Hence $P(x)$ is an even function.

(Clearly the product of two polynomial functions is also a polynomial function)

b. Outcomes assessed: E2

Marking Guidelines

Criteria	Marks
• applies alternate segment theorem to deduce EF and BC are parallel	1
• deduces equality of angles DEF and BDE	1
• deduces equality of angles DEF and DAF	1
• deduces equality of angles BDE and DAE	1

Answer

Construct EF and ED.

Applying the alternate segment theorem in circles ABC and AEF:

$\angle ABC=\angle NAC=\angle AEF$

$\therefore EF \parallel BC$ (*equal corresp.* $\angle$'s *on transversal AB*)

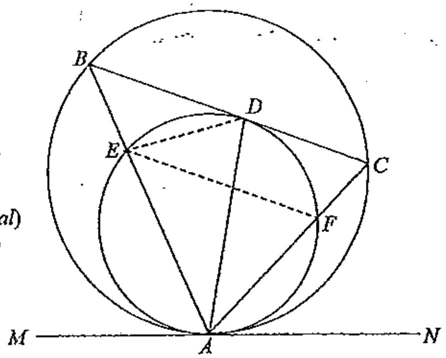
$\therefore \angle DEF=\angle BDE$ (*Alt.* $\angle$'s *within parallel lines are equal*)

$\angle DEF=\angle DAF$ ($\angle$'s *subtended by same arc DF at circumference of circle AEF are equal*)

$\angle BDE=\angle DAE$ (*Alt. segment theorem in circle AEF*)

$\therefore \angle DAF=\angle DAE$

Hence AD bisects $\angle BAC$



c. Outcomes assessed: E2

Marking Guidelines

Criteria	Marks
i • uses index laws and positive indices $a-b$, $b-c$ to deduce $a^{a-b}b^{b-c}>c^{a-c}$	1
• rearranges or uses further index laws to produce required inequality	1
ii • applies the increasing function $\ln(x)$ to both sides, preserving the direction of the inequality	1
• uses log laws to complete the proof	1

Answer

i. $a^{a-b}b^{b-c}>c^{a-b}c^{b-c}=c^{a-c}$, since $a>b>c>1$

$\therefore \quad a^{a-b}b^{b-c}>c^{a-c}$

$(a^bb^cc^c)a^{a-b}b^{b-c}>(a^bb^cc^c)c^{a-c}$

$a^ab^bc^c \quad > \quad a^bb^cc^a$

ii. Since $f(x)=\ln x$ is monotonic increasing,

$\ln(a^ab^bc^c)>\ln(a^bb^cc^a)$

$\ln a^a+\ln b^b+\ln c^c>\ln a^b+\ln b^c+\ln c^a$

$\therefore a\ln a+b\ln b+c\ln c>b\ln a+c\ln b+a\ln c$

d. Outcomes assessed: E3

Marking Guidelines

Criteria	Marks
i • factorises z^n-1, then uses the sum of n terms of a GP to deduce the required result	1
• recognises PP_k as $\lvert 1-\omega^k\rvert$ then evaluates the product using the sum of powers of 1	1
ii • shows $PP_k^2=2-2\operatorname{Re}(\omega^k)$	1
• rearranges expression for the sum of squares using properties of $\operatorname{Re}(z)$	1
• uses relationship between sum of the roots and coefficients of $z^n-1=0$ to complete the proof	1

Answer

i. $(z-1)(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1})=z^n-1$

$(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1})=\dfrac{z^n-1}{z-1}$

But $\quad 1+z+z^2+\ldots+z^{n-1}=\dfrac{z^n-1}{z-1}$

$\therefore (z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1})=1+z+z^2+\ldots+z^{n-1}$

$PP_k=\lvert 1-\omega^k\rvert$

$\therefore PP_1\times PP_2\times\ldots\times PP_{n-1}$

$=\lvert 1-\omega\rvert.\lvert 1-\omega^2\rvert.\ldots\lvert 1-\omega^{n-1}\rvert$

$=\lvert(1-\omega)(1-\omega^2)\ldots(1-\omega^{n-1})\rvert$

$=1+1+1^2+\ldots+1^{n-1}$

$=n$

ii.

$PP_k^2=\lvert 1-\omega^k\rvert^2$

$=(1-\omega^k)\overline{(1-\omega^k)}$

$=(1-\omega^k)(1-\overline{\omega^k})$

$=1+\omega^k\overline{\omega^k}-(\omega^k+\overline{\omega^k})$

$=1+\lvert\omega^k\rvert^2-2\operatorname{Re}(\omega^k)$

$=2-2\operatorname{Re}(\omega^k)$

$PP_1^2+PP_2^2+\ldots+PP_{n-1}^2=2\left\{\sum_{k=1}^{n-1}(1-\operatorname{Re}(\omega^k))\right\}$

$=2\left\{(n-1)-\sum_{k=1}^{n-1}\operatorname{Re}(\omega^k)\right\}$

$=2\{n-\operatorname{Re}(1+\omega+\omega^2+\ldots+\omega^{n-1})\}$

$=2\{n-\operatorname{Re}(0)\}$

$=2n$