STUDENT NUMBER/NAME: ....

NSW INDEPENDENT SCHOOLS

2015
Higher School Certificate
Trial Examination

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Board approved calculators may be used.
- Write using black or blue pen
- A table of standard integrals is provided at the back of the paper
- All necessary working should be shown in Question 11 – 16
- Write your student number and/or name at the top of every page

Total marks – 100

Section I - Pages 2 – 5

10 marks

Attempt Questions 1 - 10

Allow about 15 minutes for this section

Section II - Pages 6 – 11

90 marks

Attempt Questions 11 – 16

Allow about 2 hours 45 minutes for this section

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x>0$

Student name / number ____________________

Marks

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

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1 What is the value of $8e^{-2}$ correct to 3 significant figures ? 1

(A) 1·08

(B) 1·082

(C) 1·083

(D) 1·10

2 What is the focus of the parabola $x^2 = 12(y+3)$? 1

(A) $(0,-3)$

(B) $(0,0)$

(C) $(0,3)$

(D) $(0,12)$

3 Which expression is a simplification of $\dfrac{\sin(\pi - x)}{\sin(\frac{\pi}{2} - x)}$? 1

(A) $\dfrac{-\sin x}{1-\sin x}$

(B) $-\tan x$

(C) $\tan x$

(D) $\dfrac{2(\pi - x)}{\pi - 2x}$

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Marks

4 Which of the following is NOT an even function ? 1

(A) $f(x) = (x-2)^2$

(B) $f(x) = |x|$

(C) $f(x) = \cos x$

(D) $f(x) = e^x + e^{-x}$

5 What is the radius of the circle $x^2 + y^2 - 4x + 8y + 11 = 0$? 1

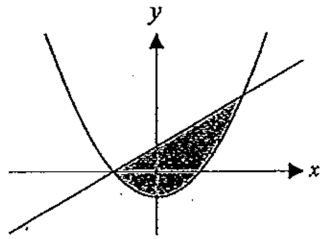
(A) 2

(B) 3

(C) 4

(D) 9

6 The diagram shows the region bounded by $x - y + 1 = 0$ and $y = x^2 - 1$. 1



Which of the following pairs of inequalities describes the shaded region in the diagram ?

(A) $x - y + 1 \geq 0$ and $y \geq x^2 - 1$

(B) $x - y + 1 \geq 0$ and $y \leq x^2 - 1$

(C) $x - y + 1 \leq 0$ and $y \geq x^2 - 1$

(D) $x - y + 1 \leq 0$ and $y \leq x^2 - 1$

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Marks

7 What is the value of $\lim_{x\to 2}\frac{x^3-8}{x-2}$? 1

(A) 0

(B) 4

(C) 8

(D) 12

8 What is the domain of the function $f(x)=\log_e x+\log_e(x-2)$? 1

(A) $x>0$

(B) $0<x<2$

(C) $x>2$

(D) *all real x*

9 What is the value of $\int \frac{\sin x}{\cos x}dx$? 1

(A) $\sec^2 x+C$

(B) $\frac{1}{2}\tan^2 x+C$

(C) $\log_e \cos x+C$

(D) $\log_e \sec x+C$

10 A water tank holds 800 litres of water. Water is let out of the tank at a rate of R litres per minute where $R=100t$ after t minutes. How long does it take the tank to empty ? 1

(A) 2 minutes

(B) 4 minutes

(C) 6 minutes

(D) 8 minutes

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Marks

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer the questions in writing booklets provided. Use a separate writing booklet for each question. In Questions 11-16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet.**

(a) Find the sum of the multiples of 6 between 1 and 400. 3

(b) Differentiate $(3-x)^4$ with respect to x. 1

(c) Differentiate $\frac{\cos x}{x^2}$ with respect to x. 2

(d) Find the equation of the tangent to the curve $y=\log_e(2x-1)$ at the point $(1,0)$. 3

(e) Evaluate $\int_{-1}^{3}\left(x^2+\frac{x}{2}\right)dx$. 3

(f) The region bounded by the curve $y=2-\sqrt{x}$ and the y axis between $y=0$ and $y=2$ is rotated about the y axis to form a solid. Find the volume of the solid in simplest exact form. 3

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Marks

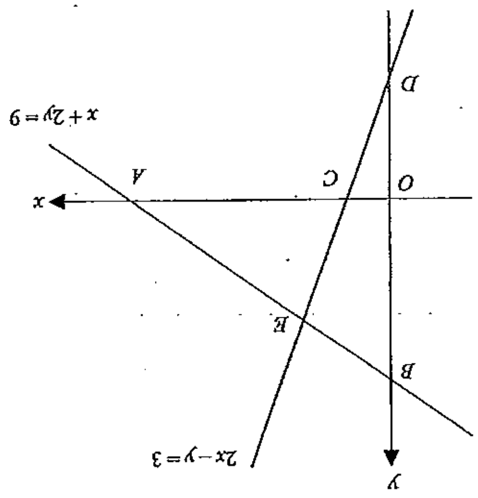
Question 12 (15 marks) **Use a separate writing booklet.**

(a) Find $\int e^x\left(e^x+1\right)dx$. 2

(b) Bag A contains 4 blue balls and 3 red balls. Bag B contains 2 blue balls and 3 red balls. Pierre chooses one ball at random from the 7 balls in bag A and puts it in bag B. He then chooses one ball at random from the 6 balls in bag B. Find the probability he chooses a red ball from bag B. 3

(c) Find the set of values of x for which the function $f(x)=xe^{-x}$ is both decreasing and concave down. 3

(d) The diagram shows a quadrilateral $OCEB$. The line $x+2y=9$ meets the x and y axes at the points A and B respectively. The line $2x-y=3$ meets the x and y axes at the points C and D respectively. E is the point of intersection of the two lines.



(i) Show that the lines are perpendicular to each other. 2

(ii) Find the coordinates of the point E. 2

(iii) Find the area of the quadrilateral $OCEB$. 3

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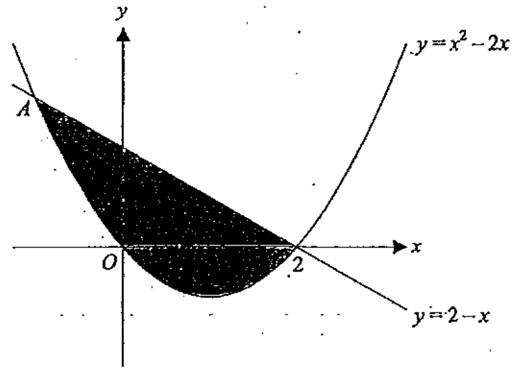
Marks

Question 13 (15 marks) **Use a separate writing booklet.**

(a) The diagram shows the graph of the parabola $y=x^2-2x$ and the line $y=2-x$. The graphs intersect at the points $(2,0)$ and A. The region bounded by the parabola and the line is shaded.



(i) Find the coordinates of A. 1

(ii) Find the area of the shaded region. 3

(b) Blaise rolls a fair die repeatedly until he first rolls a 5 or a 6.

(i) Find as a fraction in its simplest terms the probability that Blaise rolls the die exactly 8 times. 2

(ii) Find as a fraction in its simplest terms the probability that Blaise rolls the die fewer than 8 times. 3

(c) The gradient function of a curve $y=f(x)$ is given by $f'(x)=4+\frac{1}{x^2}$. 3

The curve passes through the point $(1,2)$. Find the equation of the curve.

(d) Use Simpson's Rule with 3 function values to obtain the approximation 3

$$\int_1^7\left(\frac{x-1}{6}\right)\log_e x\,dx \approx \log_e 112.$$

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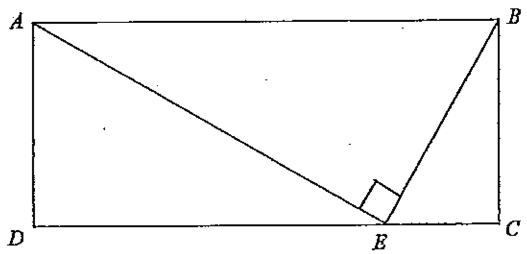
Marks

Question 14 (15 marks) **Use a separate writing booklet.**

(a) In the diagram $ABCD$ is a rectangle. E is the point on DC, where $DE > CE$, such that AE and BE are perpendicular to each other.



(i) Show $\Delta AED \parallel\!| \Delta EBC$. 3

(ii) If $AB = 8$ cm and $BC = 3$ cm, find the length of DE in simplest exact form. 3

(b) A particle is moving in a straight line. At time t seconds its displacement is x metres from a fixed point O on the line and its velocity v ms^{-1} is given by $v = 3t^2 - 2t - 1$. Initially the particle is 1 metre to the right of O.

(i) Show that the particle is at rest after 1 second. 1

(ii) Find the displacement x in terms of t. 2

(iii) Find the distance travelled by the particle in the first 2 seconds of its motion. 2

(c) Initially there are 1200 individuals in a population. After t years there are $N(t)$ individuals in the population where $N(t) = 1200\, e^{kt}$ for some constant $k > 0$. After 18 years there 3000 individuals in the population.

(i) Show that $k \approx 0 \cdot 0509$ correct to 4 decimal places. 1

(ii) Find in years and months, correct to the nearest month, the time taken for the number of individuals to increase from 1200 to 2400. 3

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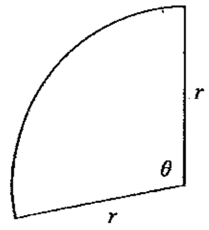
Marks

Question 15 (15 marks) **Use a separate writing booklet.**

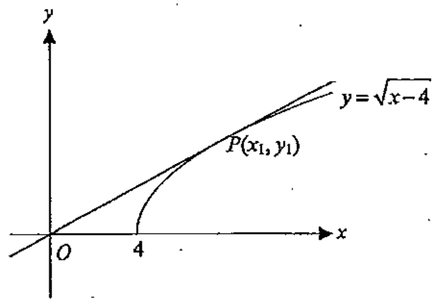
(a) In the diagram the sector has radius r cm and contains an angle θ radians. The area of the sector is 100 cm^2 .



(i) Show that the perimeter P cm of the sector is given by $P = 2r + \dfrac{200}{r}$. 2

(ii) If r and θ vary in such a way that the area of the sector remains constant at 100 cm^2 , find the values of r and θ which give the least value of P. 3

(b) The tangent to the curve $y = \sqrt{x-4}$ at the point $P(x_1, y_1)$ passes through the origin $O(0,0)$. By considering the gradient of OP in two different ways, find the value of x_1 . 3



(c)(i) Solve the equation $1 - 2\cos x = 0$ for $0 \le x \le 2\pi$. 2

(ii) ~~Sketch the graph of the curve $y = 1 - 2\cos x$ for $0 \le x \le 2\pi$ showing clearly~~ the coordinates of the endpoints and the maximum turning point. 2

(iii) Find in simplest exact form the area of the region bounded by the curve $y = 1 - 2\cos x$ and the x axis between $x = 0$ and $x = \pi$. 3

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Marks

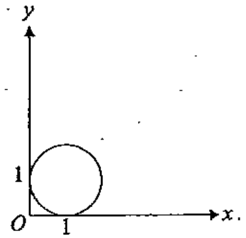
Question 16 (15 marks) **Use a separate writing booklet.**

(a) Ozzie takes out a loan of \$400 000. The loan is to be repaid in equal monthly repayments of \$3600. Reducible interest is charged at 6% p.a. calculated monthly. Let $\$A_n$ be the amount owing after the n^{th} repayment.

(i) Show that $A_3 = 400\,000(1\cdot005)^3 - 3600(1 + 1\cdot005 + 1\cdot005^2)$. **2**

(ii) Find correct to the nearest month the time taken for Ozzie to repay the loan. **4**

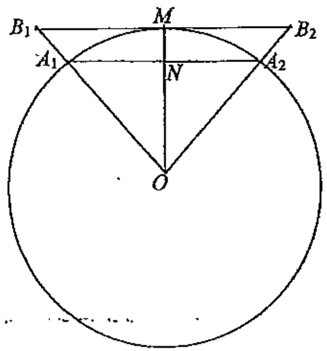
(b) In the diagram the circle with centre $(1,1)$ and radius 1 touches both the x and y axes.



(i) Show that the line $3x + 4y - 12 = 0$ is tangent to the circle. **2**

(ii) Find the radius r of the other circle with centre in the first quadrant that also touches both the x and y axes and for which the line $3x + 4y - 12 = 0$ is also a tangent to the circle. **2**

(c) In the diagram O is the centre of a circle with radius r. A_1A_2 is one side of a regular n-sided polygon which lies inside the circle so that the vertices of the polygon lie on the circle. B_1B_2 is one side of a regular n-sided polygon which lies outside the circle so that the sides of the polygon touch the circle. M is the point of contact of B_1B_2 with the circle. The line OM cuts A_1A_2 at the point N, and OM is perpendicular to both A_1A_2 and B_1B_2.



(i) Show that $Area\ \Delta OA_1A_2 = r^2 \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)$. Find a similar expression for $Area\ \Delta OB_1B_2$. **3**

(ii) Hence show $\dfrac{Area\ of\ outside\ polygon}{Area\ of\ inside\ polygon} = \sec^2\left(\frac{\pi}{n}\right)$. **2**

Section 1 Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1.	A	$8e^{-2} \approx 1\cdot08268... \approx 1\cdot08$ (to 3 significant figures)	P3
2.	B	Parabola $x^2 = 12(y+3)$ has vertex $(0,-3)$ and focal length 3. Focus is $(0,0)$	P4
3.	C	$\frac{\sin(\pi - x)}{\sin(\frac{\pi}{2} - x)} = \frac{\sin x}{\cos x} = \tan x$	P3
4.	A	For $f(x) = (x-2)^2$, $f(-1) = 9$ and $f(1) = 1$. $\therefore f(-1) \neq f(1)$	P5
5.	B	$x^2 + y^2 - 4x + 8y + 11 = 0 \Rightarrow (x-2)^2 + (y+4)^2 = 9$. Circle has radius 3.	P4
6.	A	(0, 0) lies in the region and satisfies both $x - y + 1 \geq 0$ and $y \geq x^2 - 1$	P5
7.	D	$\lim_{x\to2} \frac{x^3-8}{x-2} = \lim_{x\to2}(x^2+2x+4) = 12$	H5
8.	C	Domain is $x > 0$ *and* $x > 2$, giving $x > 2$	H3
9.	D	$\int \frac{\sin x}{\cos x} dx = -\log_e \cos x = \log_e\left(\frac{1}{\cos x}\right) = \log_e \sec x$	H5
10.	B	Amount of water let out in t minutes is V litres where $V = 50t^2 + c$, for some constant c. Initially $V = 0$ $\therefore c = 0$ and $V = 50t^2$. Tank is empty when 800L has been let out. $800 = 50t^2$ gives $t^2 = 16$. $\therefore t = 4$	H8

Section II

Question 11

a. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• identifies the sum as an arithmetic series with first term 6 and last term 396	1
• finds the number of terms in the series	1
• calculates the sum	1

Answer

$400 = 6 \times 66 + 4$ $\quad \therefore 6+12+18+...+396 = S_{66}$ for an AP with common difference 6.

$\therefore 6+12+18+...+396 = 33(6+396) = 13\,266$

b. Outcomes assessed: P6

Marking Guidelines

Criteria	Marks
• uses the chain rule to differentiate the expression	1

Answer

$$\frac{d}{dx}(3-x)^4 = -4(3-x)^3$$

Q11 (cont)

c. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• applies the quotient rule	1
• simplifies the derivative	1

Answer

$$\frac{d}{dx}\left(\frac{\cos x}{x^2}\right) = \frac{-\sin x \cdot x^2 - \cos x \cdot 2x}{x^4} = \frac{-x(x\sin x + 2\cos x)}{x^4} = \frac{-(x\sin x + 2\cos x)}{x^3}$$

d. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• finds the derivative	1
• finds the gradient of the tangent	1
• finds the equation of the tangent	1

Answer

$y = \log_e(2x-1)$ $\quad \therefore \frac{dy}{dx} = \frac{2}{2x-1} = 2$ when $x = 1$. Hence tangent at $(1,0)$ has gradient 2 and equation $y = 2x - 2$.

e. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• finds the primitive function	1
• substitutes to obtain a numerical expression	1
• evaluates	1

Answer

$$\int_{-1}^{3}\left(x^2 + \frac{x}{2}\right)dx = \left[\tfrac{1}{3}x^3 + \tfrac{1}{4}x^2\right]_{-1}^{3} = \tfrac{1}{3}\{3^3 - (-1)^3\} + \tfrac{1}{4}\{3^2 - (-1)^2\} = \tfrac{28}{3} + \tfrac{8}{4} = 11\tfrac{1}{3}$$

f. Outcomes assessed: H8

Marking Guidelines

Criteria	Marks
• expresses the volume as a definite integral in terms of y	1
• finds the primitive function	1
• evaluates the definite integral as required	1

Answer

$$V = \int_0^2 \pi x^2\, dy = \int_0^2 \pi(2-y)^4\, dy = -\tfrac{\pi}{5}\left[(2-y)^5\right]_0^2 = -\tfrac{\pi}{5}\{0 - 2^5\} = \tfrac{32\pi}{5}$$

Ans. $\frac{32\pi}{5}$ cu. units

Question 12

a. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• expands the integrand or recognises the pattern $f'.f^n$	1
• writes the primitive function	1

Answer

$$\int e^x(e^x+1)dx = \int(e^{2x}+e^x)dx = \tfrac{1}{2}e^{2x}+e^x+c \quad \text{or} \quad \int e^x(e^x+1)dx = \tfrac{1}{2}(e^x+1)^2+c$$

b. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• finds an expression for the probability of selecting a red from A then a red from B	1
• finds an expression for the probability of selecting a blue from A then a red from B	1
• adds and simplifies to obtain the required probability	1

Answer

$P(\textit{red from A then a red from B}) = \frac{3}{7}\times\frac{4}{6}$

$P(\textit{blue from A then a red from B}) = \frac{4}{7}\times\frac{3}{6}$

$\therefore P(\textit{a red from B}) = \frac{3}{7}\times\frac{4}{6}+\frac{4}{7}\times\frac{3}{6} = \frac{4}{7}$

c. Outcomes assessed: H6

Marking Guidelines

Criteria	Marks
• applies the product rule to find the first derivative	1
• finds the second derivative	1
• deduces the set of values for x	1

Answer

$f(x)=xe^{-x}$ $\quad f'(x)=1.e^{-x}+x.(-e^{-x})=(1-x)e^{-x}$ $\quad \therefore f'(x)<0$ for $x>1$

$f''(x)=-e^{-x}-(1-x)e^{-x}=(x-2)e^{-x}$ $\quad \therefore f''(x)<0$ for $x<2$

Hence function is decreasing and concave down for $1<x<2$

Q12 (cont)

d. Outcomes assessed: P4, H5

Marking Guidelines

Criteria	Marks
i • finds the gradients of both lines	1
• verifies that the product of the gradients is −1	1
ii • solves simultaneous equations to find one coordinate of E	1
• finds the second coordinate of E	1
iii • finds the coordinates of at least two of the points A, B, C, D	1
• writes an expression for the required area in terms of the area of two simpler figures	1
• calculates the required area	1

Answer

i. Line $x+2y=9$ has gradient/intercept form $y=-\frac{1}{2}x+\frac{9}{2}$ and gradient $-\frac{1}{2}$.
Line $2x-y=3$ has gradient/intercept form $y=2x-3$ and gradient 2.
Since $-\frac{1}{2}\times 2=-1$, the lines are perpendicular.

ii. At point of intersection E, solving simultaneously: (1) $2x-y=3$, (2) $x+2y=9$ $\quad 2\times(1)+(2) \Rightarrow 5x=15$, $x=3$

Substituting $x=3$ in (1) gives $y=3$. E has coordinates $(3,3)$.

iii. On $x+2y=9$, x and y intercepts are $A(9,0)$ and $B(0,4\cdot5)$.
On $2x-y=3$, x and y intercepts are $C(1\cdot5,0)$ and $D(0,-3)$.

$\textit{Area } OCEB = \textit{Area } \Delta OAB - \textit{Area } \Delta CAE$
$=\frac{1}{2}\times9\times4\cdot5-\frac{1}{2}\times(9-1\cdot5)\times3$
$=9$

or

$\textit{Area } OCEB = \textit{Area } \Delta DBE - \textit{Area } \Delta DOC$
$=\frac{1}{2}\times7\cdot5\times3-\frac{1}{2}\times1\cdot5\times3$
$=9$

Question 13

a. Outcomes assessed: P4, H8

Marking Guidelines

Criteria	Marks
i • solves the equations of line and parabola simultaneously to find the coordinates of A	1
ii • finds an expression for the area as a definite integral	1
• finds the primitive function	1
• evaluates to find the area	1

Answer

i. At A, $\left.\begin{array}{l} y=x^2-2x \\ y=2-x \end{array}\right\} \Rightarrow \begin{array}{l} x^2-2x=2-x \\ x^2-x-2=0 \\ (x-2)(x+1)=0 \end{array}$ But $x\neq2$ $\therefore \left.\begin{array}{l} x=-1 \\ y=3 \end{array}\right\}$. Hence $A(-1,3)$.

ii. Shaded area is given by $\int_{-1}^{2}\{(2-x)-(x^2-2x)\}dx=\int_{-1}^{2}(2+x-x^2)dx$

where $\int_{-1}^{2}(2+x-x^2)dx=\left[2x+\frac{1}{2}x^2-\frac{1}{3}x^3\right]_{-1}^{2}=2\{2-(-1)\}+\frac{1}{2}(4-1)-\frac{1}{3}\{8-(-1)\}=4\frac{1}{2}$

Hence shaded area is $4\frac{1}{2}$ sq.units

Q13 (cont)

b. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
i • finds the probabilities of the events '*rolls 5 or 6*' and its complement	1
• calculates the required probability	1
ii • interprets '*fewer than* 8 *rolls*' as complement of '*no 5's or 6's on first 7 rolls*'	1
• writes a numerical expression for this probability	1
• evaluates this probability as required	1

Answer

i. On each roll, $P(rolls\ 5\ or\ 6)=\frac{2}{6}=\frac{1}{3}$. Hence $P(does\ not\ roll\ 5\ or\ 6)=\frac{2}{3}$.

$P(Blaise\ rolls\ exactly\ 8\ times)=\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)=\frac{128}{6561}$

ii. $P(fewer\ than\ 8\ rolls)=1-P(no\ 5's\ or\ 6's\ on\ first\ 7\ rolls)=1-\left(\frac{2}{3}\right)^7=\frac{2059}{2187}$

c. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
• finds primitive function with constant c	1
• uses given coordinates to evaluate c	1
• writes the equation of the curve	1

Answer

$f'(x)=4+x^{-2}\quad \therefore f(x)=4x-x^{-1}+c$. But $f(1)=2 \Rightarrow 4-1+c=2\quad \therefore c=-1$

Hence curve has equation $y=4x-\dfrac{1}{x}-1$

d. Outcomes assessed: H3

Marking Guidelines

Criteria	Marks
• finds the three function values	1
• applies Simpson's rule to find an expression for the definite integral	1
• applies appropriate logarithm laws to simplify this approximation	1

Answer

x	1	4	7
f	0	$\frac{1}{2}\log_e 4$	$\log_e 7$
multiplier	×1	×4	×1

$h=3$

$f(x)=\frac{1}{6}(x-1)\log_e x$

$\int_1^7 f(x)\,dx\approx\frac{3}{3}\{0+2\log_e 4+\log_e 7\}=\log_e(4^2\times 7)=\log_e 112$

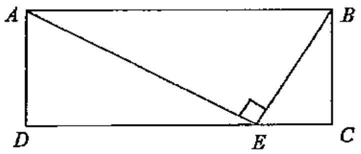
Question 14

a. Outcomes assessed: P4, H2, H5

Marking Guidelines

Criteria	Marks
i • notes the equal right angles at D and C, quoting the appropriate rectangle property	1
• explains why $\angle AED$ and $\angle BEC$ are complementary	1
• deduces the second pair of equal angles and completes the proof	1
ii • uses sides in proportion to write a quadratic equation for DE	1
• completes the square or uses the quadratic formula	1
• chooses the correct root and gives simplest exact form	1

Answer



i. In ΔAED, ΔEBC

$\angle ADE=\angle ECB=90°$ (*Vertex angles of rectangle ABCD are* 90°)

Also $\angle AED+\angle BEC=90°$ (*straight angle DEC is* 180°)

and $\angle EBC+\angle BEC=90°$ ($\angle$ *sum of* ΔEBC *is* 180°)

$\therefore \angle AED=\angle EBC$

Then $\angle EAD=\angle BEC$ ($\angle$ *sum of each* Δ *is* 180°)

$\therefore \Delta AED\ |||\ \Delta EBC$ (*equiangular*)

ii. $\dfrac{DE}{CB}=\dfrac{AD}{EC}$ (*sides of similar triangles are in proportion*)

$\dfrac{DE}{3}=\dfrac{3}{8-DE}$ (*opposite sides of a rectangle are equal*)

$DE(8-DE)=9$ Hence $4<DE<8$ is a root of the equation $x^2-8x=-9$

$(x-4)^2=7$

$\therefore DE=\left(4+\sqrt{7}\right)$ cm

Q14 (cont)

b. Outcomes assessed: P4, H5, H9

Marking Guidelines

Criteria	Marks
i • shows $v=0$ when $t=1$	1
ii • finds an expression for x by integration	1
• uses the initial conditions to evaluate the constant of integration	1
iii • finds the position of the particle when it changes its direction of travel	1
• uses the path of travel to find the distance over the first two seconds	1

Answer

i. $t=1 \Rightarrow v=3t^2-2t-1=3-2-1=0$. Hence particle is at rest after 1 second.

ii. $x=t^3-t^2-t+c$, and $x=1$ when $t=0 \Rightarrow c=1$ $\quad \therefore x=t^3-t^2-t+1$

iii. $v=(3t+1)(t-1)$ $\therefore v<0$ for $0\le t<1$, and $v>0$ for $t>1$.
Hence particle changes direction at $t=1$ where $x=1-1-1+1=0$. When $t=2,\ x=8-4-2+1=3$.
Particle starts at $x=1$, travels left to $x=0$, then right to $x=3$. Distance travelled is 4 metres.

c. Outcomes assessed: H4, H5

Marking Guidelines

Criteria	Marks
i • substitutes and evaluates k.	1
ii • evaluates kt	1
• evaluates t	1
• interprets the answer to give the time in years and months	1

Answer

i. $N(t)=1200e^{kt}$. $\quad N(18)=3000 \Rightarrow 3000=1200e^{18k}$ $\quad \therefore e^{18k}=\frac{5}{2}$

$18k=\ln\frac{5}{2}$

$k=(\ln\frac{5}{2})\div 18 \approx 0\cdot 0509$

ii. $N(t)=2400 \Rightarrow 2400=1200e^{kt}$ $\quad \therefore e^{kt}=2$

$kt=\ln 2$ $\quad \therefore t=\ln 2 \div k \approx 13\cdot 61647...$

Time taken is 13 years and 7 months (to the nearest month).

Question 15

a. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
i • uses the formula for area of a sector to obtain the relationship between r and θ .	1
• uses this relationship to express P in terms of r alone.	1
ii • derives P with respect to r and deduces a stationary value at $r=10$	1
• checks this stationary value is a minimum	1
• finds the corresponding value of θ	1

Answer

i. $A=\frac{1}{2}r^2\theta \Rightarrow r^2\theta=200$ $\quad \therefore r\theta=\frac{200}{r}$ $\quad \therefore P=2r+r\theta=2r+\frac{200}{r}$

ii. $\frac{dP}{dr}=2-\frac{200}{r^2}$ $\quad \therefore \frac{dP}{dr}=0$ for $r=10$, where $\frac{d^2P}{dr^2}=\frac{400}{r^3}>0$. $\quad \therefore$ Minimum P for $\begin{cases} r=10 \\ \theta=2 \end{cases}$

b. Outcomes assessed: P4, P6

Marking Guidelines

Criteria	Marks
• uses differentiation to find the gradient at P	1
• finds the gradient of OP from the coordinates of P then expresses this in terms of x_1 alone	1
• writes and solves an equation for x_1	1

Answer

$y=(x-4)^{\frac{1}{2}}$

$\frac{dy}{dx}=\frac{1}{2}(x-4)^{-\frac{1}{2}}$

Hence gradient of the tangent at $P(x_1,y_1)$ is $\frac{1}{2\sqrt{x_1-4}}$.

Also gradient of OP is $\frac{y_1}{x_1}=\frac{\sqrt{x_1-4}}{x_1}$. $\quad \therefore \frac{1}{2\sqrt{x_1-4}}=\frac{\sqrt{x_1-4}}{x_1}$

$\therefore x_1=2(x_1-4)$ and $x_1>4$

$x_1=2x_1-8$

$x_1=8$

Q15 (cont)

c. Outcomes assessed: H5, H8

Marking Guidelines

Criteria	Marks
i • finds one solution	1
• finds the second solution	1
ii • sketches curve with correct shape and intercepts on the coordinate axes	1
• shows coordinates of maximum turning point and right-hand endpoint	1
iii • expresses the area in terms of definite integrals	1
• finds the primitive function	1
• evaluates exact area	1

Answer

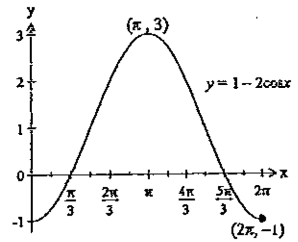
i.

$1-2\cos x=0,\quad 0\le x\le 2\pi$

$\cos x=\frac{1}{2}$

$x=\frac{\pi}{3},\ \frac{5\pi}{3}$

ii



iii. Area is A sq.units where

$$A=-\int_0^{\frac{\pi}{3}}\left(1-2\cos x\right)dx+\int_{\frac{\pi}{3}}^{\pi}\left(1-2\cos x\right)dx$$

$$=-\left[x-2\sin x\right]_0^{\frac{\pi}{3}}+\left[x-2\sin x\right]_{\frac{\pi}{3}}^{\pi}$$

$$=-\left(\frac{\pi}{3}-2\frac{\sqrt{3}}{2}\right)+\left\{\left(\pi-\frac{\pi}{3}\right)-2\left(0-\frac{\sqrt{3}}{2}\right)\right\}$$

$$=\frac{\pi}{3}+2\sqrt{3}$$

Question 16

a. Outcomes assessed: H1, H2, H4, H5

Marking Guidelines

Criteria	Marks
i • writes expression for A_1, then for A_2	1
• writes expression for A_3 in terms of A_2, then as required	1
ii • generalises to obtain expression for A_n	1
• sums the geometric progression and simplifies the resulting expression for A_n	1
• solves $A_n=0$ for n	1
• interprets this result to give time to repay loan, remembering to round up to next whole month	1

Answer

i. Monthly interest rate is $0\cdot5\%$. $\therefore r=0\cdot005$

$A_1=P\left(1\cdot005\right)-3600=400\,000\left(1\cdot005\right)-3600$

$A_2=A_1\left(1\cdot005\right)-3600=400\,000\left(1\cdot005\right)^2-3600\left(1+1\cdot005\right)$

$A_3=A_2\left(1\cdot005\right)-3600=400\,000\left(1\cdot005\right)^3-3600\left(1+1\cdot005+1\cdot005^2\right)$

Q16 a (cont)

ii. Generalising this result, $A_n=400\,000\left(1\cdot005\right)^n-3600\left(1+1\cdot005+1\cdot005^2+\ldots+1\cdot005^{n-1}\right)$

$$A_n=400\,000\left(1\cdot005\right)^n-3600\left(\frac{1\cdot005^n-1}{1\cdot005-1}\right)$$

$$=400\,000\left(1\cdot005\right)^n-720\,000\left(1\cdot005^n-1\right)$$

$$=720\,000-320\,000\left(1\cdot005\right)^n$$

$$A_n=0\Rightarrow 320\,000\left(1\cdot005\right)^n=720\,000$$

$$\left(1\cdot005\right)^n=\frac{9}{4}$$

$$n\log_e 1\cdot005=\log_e\frac{9}{4}$$

$$\therefore n=\log_e\frac{9}{4}\div\log_e 1\cdot005\approx162\cdot59117\ldots$$

Hence loan is repaid after 13 years and 7 months

b. Outcomes assessed: H5

Marking Guidelines

Criteria	Marks
i • finds the perpendicular distance from the circle centre to the line	1
• compares this distance with the radius of the circle to deduce the result	1
ii • expresses the perpendicular distance from the centre of the circle to the line in terms of r	1
• writes and solves an equation for r	1

Answer

i. Perpendicular distance from $(1, 1)$ to the line $3x+4y-12=0$ is $\frac{\left|3+4-12\right|}{\sqrt{3^2+4^2}}=\frac{\left|-5\right|}{5}=1$

Since this distance is equal to the radius of the circle, the line must be tangent to the circle.

ii. Since the second circle touches both the x and y axes, its centre must have coordinates $(r\,,\ r)$ where r is the radius of the circle.

The perpendicular distance d from $(r\,,\ r)$ to $3x+4y-12=0$ is given by $d=\frac{\left|3r+4r-12\right|}{\sqrt{3^2+4^2}}=\frac{\left|7r-12\right|}{5}$.

The line is tangent to the circle if $d=r$. $\therefore\left|7r-12\right|=5r$

$\therefore 7r-12=5r$ or $7r-12=-5r$

$2r=12$ $\qquad$ $12r=12$

$r=6$ $\qquad$ $r=1$

Hence the radius of the second circle is 6.

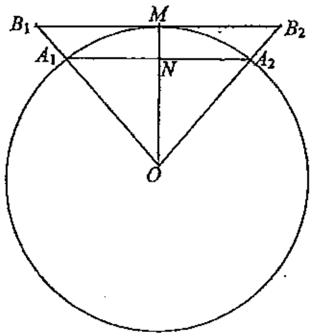
16 (cont)

c. **Outcomes assessed: H5**

Marking Guidelines

Criteria	Marks
i • uses symmetry to deduce the size of angle A_1ON and hence the lengths of ON and A_1A_2	1
• deduces the required area of triangle OA_1A_2	1
• finds an expression for B_1B_2 and hence for the area of triangle OB_1B_2	1
ii • writes the required ratio using the expressions from i.	1
• simplifies using appropriate trigonometric identities	1

Answer



i. $\angle B_1OB_2 = \angle A_1OA_2 = \frac{2\pi}{n}$ and OM is an axis of symmetry for both isosceles triangles OA_1A_2 and OB_1B_2.

$\therefore \angle A_1ON = \frac{\pi}{n}$ and in right-angled ΔA_1ON, $ON = r\cos\frac{\pi}{n}$ and $A_1A_2 = 2A_1N = 2r\sin\frac{\pi}{n}$

Hence $Area\ \Delta OA_1A_2 = \frac{1}{2}\times A_1A_2 \times ON = r^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$

In ΔOB_1B_2, $OM = r$ and $\frac{1}{2}B_1B_2 = MB_1 = r\tan\frac{\pi}{n}$. Hence $Area\ \Delta OB_1B_2 = \frac{1}{2}\times B_1B_2\times OM = r^2\tan\frac{\pi}{n}$

ii. $$\frac{Area\ of\ outside\ polygon}{Area\ of\ inside\ polygon} = \frac{n\times Area\ \Delta OB_1B_2}{n\times Area\ \Delta OA_1A_2} = \frac{nr^2\tan\frac{\pi}{n}}{nr^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}} = \frac{\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}\cos^2\frac{\pi}{n}} = \frac{1}{\cos^2\frac{\pi}{n}} = \sec^2\left(\frac{\pi}{n}\right)$$