Name: ..... Maths Class: .....

SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC COURSE

Extension 1 Mathematics

Assessment 3
June 2013

TIME ALLOWED: **70 minutes**

Instructions:

- ***Start each question on a new page.***
- Write your name and class at the top of this page, and on all your answer sheets.
- Hand in your answers attached to the rear of this question sheet.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks indicated within each question are a guide only and may be varied at the time of marking
- It is suggested that you spend no more than 5 minutes on Part A.
- Approved calculators may be used.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n\neq -1;\ x\neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\ a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x>0$

PART A: (5 Marks)

Answers to these multiple choice should be completed on the multiple choice answer sheet supplied with your answer booklet.

All questions are worth 1 mark

1	$\frac{d}{dx} ln(\frac{x+1}{2-x}) =$ A. $\frac{3}{(x+1)(2-x)}$ B. $\frac{1-2x}{(x+1)(2-x)}$ C. $\frac{1-x}{(x+1)(2-x)}$ D. $\frac{2x-1}{(x+1)(2-x)}$
2	An indefinite integral of $\frac{1}{2}(e^x + e^{-x})$ is: A. $\frac{1}{2}(e^x + e^{-x})$ B. $-\frac{1}{2}(e^x + e^{-x})$ C. $\frac{1}{2}(e^x - e^{-x})$ D. $-\frac{1}{2}(e^x - e^{-x})$
3	The indefinite integral of $\frac{1}{\sqrt{9-x^2}}$ is: A. $\frac{1}{3}sin^{-1}\frac{x}{3} + k$ B. $sin^{-1}\frac{x}{3} + k$ C. $3sin^{-1}\frac{x}{3} + k$ D. $\frac{1}{3}sin^{-1}3x + k$
4	The value of $sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is: A. $\frac{\pi}{3}$ B. $\frac{\pi}{6}$ C. $-\frac{\pi}{6}$ D. $-\frac{\pi}{3}$
5	The curve $y = \frac{1}{\sqrt{16+x^2}}$ between the lines $x = 0$ and $x = 4$ is rotated about the x-axis. Its volume is given by: A. $\frac{\pi}{16}$ B. $\frac{\pi}{4}$ C. $\frac{\pi^2}{16}$ D. $\frac{\pi^2}{4}$

PART B

QUESTION 1: (8 marks) **Marks**

(a) Find $\frac{d}{dx} tan^{-1}\sqrt{x}$ (simplify your answer) **2**

(b) (i) Find $\frac{d}{dx}(e^{-x^2})$ **1**

(ii) Hence find $\frac{d^2}{dx^2}(e^{-x^2})$ **1**

(c) Evaluate $\int_0^{\sqrt{2}} \frac{2}{\sqrt{4-x^2}}\, dx$ **2**

(d) If $y = tan^{-1}x$ find an expression for $\sin 2y$. **2**

QUESTION 2: (Start a new page) (8 marks) **Marks**

(a) Find $sin^{-1}\left[cos(\frac{3\pi}{4})\right]$ **1**

(b) You are given the function $f(x) = (x+2)^2$

(i) State the Domain and Range of $f(x)$ and sketch the curve **2**

(ii) Find the largest possible domain of $f(x)$, containing the point (0, 4) for an inverse function $y = f^{-1}(x)$ to exist. **1**

(iii) Find the inverse function $y = f^{-1}(x)$ from part (ii) above and give its Domain and Range **3**

(iv) Sketch $y = f^{-1}(x)$ **1**

QUESTION 3: (Start a new page) (8 marks)

Marks

(a) Find the exact area beneath the curve $y = \frac{e^x}{1+e^x}$, above the x-axis, and between the lines $x = 0$ and $x = 1$ — 2

Give your answer in simplest terms.

(b) A radio-active substance decomposes, and the mass present (M) after t years from a certain date is given by

$M = M_0 e^{-kt}$ where M_0 and k are constants

(i) Show that this is a solution to $\frac{dM}{dt} = -kM$ — 1

(ii) If the initial mass is 100 gm and the mass after 2 years is 80 gm, find the value of k to 2 dec. places. — 2

(iii) Find the number of years taken for the mass to halve (called the *half life* of the substance). Give your answer to 1 decimal place. — 2

(iv) Sketch the graph of $M = M_0 e^{-kt}$ using the vertical axis as M and the horizontal axis as t. ***Show only keypoints.*** — 1

QUESTION 4: (Start a new page) (8 marks)

Marks

(a) (i) Find $\frac{d}{dx}(sin^{-1}x + cos^{-1}x)$ — 1

(ii) Hence find the exact value of $sin^{-1}x + cos^{-1}x$ — 2

You must justify your answer NOT just state it.

(b) Evaluate $\int 2^x dx$ — 1

(c) The pressure P gm/cm^3 on a mass of gas, of volume vcm^3 is given by the formula — 4

$$Pv = 1500.$$

If the volume is increasing at the rate of $10cm^3/sec$, find the rate at which the pressure is decreasing when the volume is 30 cm^3

QUESTION 5 : (Start a new page) (8 marks)

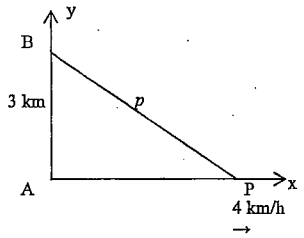
Marks

(a) (i) Find $\frac{d}{dx}(ln(sinx))$ 1

(ii) Hence, or otherwise, find $\int cot3x\ dx$ 1

(b) Find $\int \frac{dx}{9+4x^2}$ 2

(c) A person, P, is walking directly east from a point A at a speed of 4 km/h (ie $\frac{dx}{dt} = 4$) and is being watched by an observer at a point B, which is 3 km due north of A, as shown below: 4



The distance between the observer and the walker is given as p km.

Find the rate of change of p when P has walked 4 km?

QUESTION 6: (Start a new page) (8 marks)

Marks

(a) Show that the curve $y = \frac{e^{-x}}{1+x^2}$ is decreasing for all x, except $x = -1$ 3

(b) The population of seals on an island is increasing at a variable rate, and the number of seals *(P)* at any time t, is given by

$$P = A(1 - e^{-kt}), \text{ where } A \text{ and } k \text{ are constants}$$

(i) Show that $\frac{dP}{dt} = k(A - P)$ 1

(ii) Show that the maximum seal Population that the island can accommodate is A. 1

(iii) If one quarter of the maximum population that the island can hold is reached after 5 hours, what fraction is populated after another 5 hours? 3

End of Examination

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PART A

Completely fill the response oval representing the most correct answer.

1. A ● B ○ C ○ D ○ ✓
2. A ○ B ○ C ● D ○ ✓
3. A ○ B ● C ○ D ○ ✓
4. A ○ B ○ C ○ D ● ✓
5. A ○ B ○ C ● D ○ ✓

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Question 1

a) $\frac{d}{dx}(\tan^{-1}\sqrt{x}) = \frac{f'(x)}{1+[f(x)]^2}$

$f(x) = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{\frac{1}{2\sqrt{x}}}{1+(\sqrt{x})^2}$

$= \frac{1}{2\sqrt{x}} \div 1+x$

$= \frac{1}{2\sqrt{x}} \times \frac{1}{1+x}$

$= \frac{1}{2\sqrt{x}(1+x)}$ // ✓✓

b) i) $\frac{d}{dx}(e^{-x^2}) = -2x.e^{-x^2}$ ✓

ii) $\frac{d^2}{dx^2}(e^{-x^2}) = -2e^{-x^2} + 4x^2e^{-x^2}$ $u = -2x$ $v = e^{-x^2}$

$u' = -2$ $v' = -2x.e^{-x^2}$

$= 2e^{-x^2}(-1+2x^2)$ // ✓

c) $\int_0^{\sqrt{2}} \frac{2}{\sqrt{4-x^2}}dx = 2\int_0^{\sqrt{2}} \frac{dx}{\sqrt{4-x^2}}$

$= 2\left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{2}}$

$= 2\left[\left(\frac{\pi}{4}-0\right)\right]$

$= \frac{\pi}{2}$ // ✓✓

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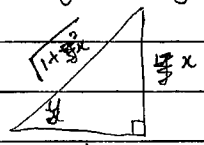
d) $y = \tan^{-1} x$

$x = \tan y$

$\sin 2y = 2\sin y . \cos y = 2 \times \frac{x}{\sqrt{1+x^2}} \times \frac{1}{\sqrt{1+x^2}}$



$= \frac{2x}{1+x^2}$

$\sin y = \frac{x}{\sqrt{x^2+1}}$

$\cos y = \frac{1}{\sqrt{x^2+1}}$

$2\sin y . \cos y$

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Question 2

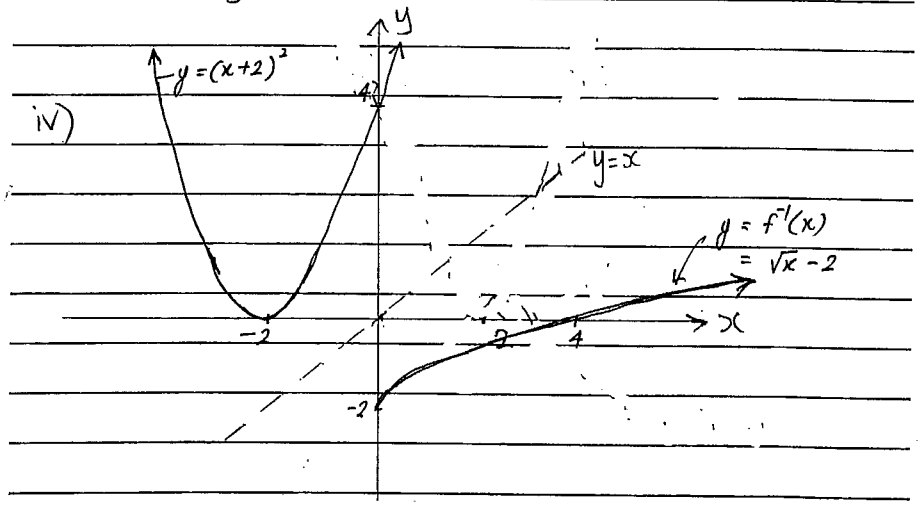
a) $\sin^{-1}\left[\cos\left(\frac{3\pi}{4}\right)\right] = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ ✓

$= -\frac{\pi}{4}$ //

b) i) D: all real x ✓

R: $y \geq 0$

iv)



ii) D: ~~$x > 0$~~ ✗ $x \geq -2$

iii) $y = (x+2)^2$

$x = (y+2)^2$

~~$x = y^2 + 4y + 4$~~ $y + 2 = \pm\sqrt{x}$ ✓ D: $x \geq 0$ ✓

Equation: $y = +\sqrt{x} - 2$ R: ~~y~~ ~~$y < 4$~~ ✗

$y \geq -2$

3

4

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Question 3

a) $\int_0^1 \frac{e^x}{1+e^x} = \left[\ln(1+e^x)\right]_0^1$

$= \left[\ln(1+e) - \ln(2)\right]$

$= \ln\left(\frac{1+e}{2}\right)$ //

b) i) $M = M_0 e^{-kt}$

$\frac{dM}{dt} = -k.M_0 e^{-kt}$

$= -kM$ as reqd. //

ii) ~~TOO EASY~~

$80 = 100e^{-2k}$

$e^{-2k} = \frac{4}{5}$

$\ln(e^{-2k}) = \ln(\frac{4}{5})$

$-2k(\ln e) = \ln(\frac{4}{5})$

$k = \frac{\ln(\frac{4}{5})}{-2}$

$= 0.11$ //

iii) $50 = 100e^{-kt}$

$\frac{1}{2} = e^{-kt}$

$\ln(\frac{1}{2}) = \ln(e^{-kt})$

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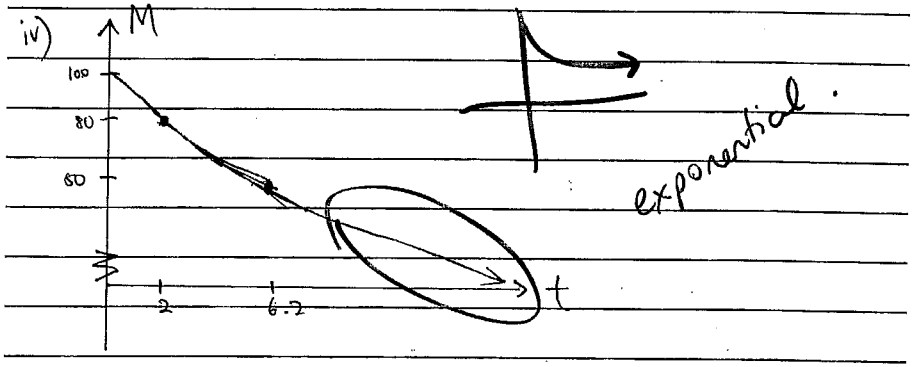
$-kt = \ln(\frac{1}{2})$

$t = \frac{\ln(\frac{1}{2})}{-k}$

$= 6.2$ years //

iv)



exponential.

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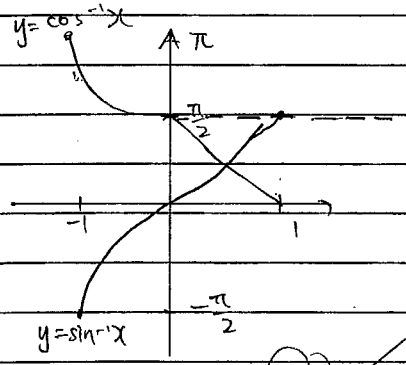
Question 4

a) i) $\frac{d}{dx}(\sin^{-1}x + \cos^{-1}x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$

$= 0$

$= 0$

ii)



$\therefore \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

you didn't state how you came to $\frac{\pi}{2}$ by substituting $x = 0$

b) $\int 2^x dx = \frac{2^x}{\ln 2} + C$

c) Want $\frac{dP}{dt}$

$P = \frac{1500}{V} = 1500V^{-1}$

$\frac{dP}{dV} = -1500V^{-2} = -\frac{1500}{V^2}$

$\frac{dV}{dt} = 10$

$\therefore \frac{dP}{dt} = \frac{dP}{dV} \times \frac{dV}{dt}$

$= -\frac{1500}{V^2} \times 10$

$= -\frac{1500}{30 \times 30} \times 10$ when $V = 30\ cm^3$

$= -\frac{50}{3} = -16\frac{2}{3}\ \frac{gm/cm^3}{sec.}$

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$PV = 1500$

$P = \frac{1500}{V}$

$= 1500V^{-1}$

$\frac{dP}{dV} = \frac{-1500}{V^2}$

$\frac{dP}{dt} = \frac{dP}{dV} \cdot \frac{dV}{dt}$

$= \frac{-1500}{V^2} 10$ sub $V = 30$

$= -\frac{5}{3} \times 10$

$= -\frac{50}{3}$

$\therefore$ Decreasing at a rate of $\frac{50}{3}\ gm/cm^3$

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Question 5

a) i) $\frac{d}{dx}(\ln(\sin x)) = \frac{+\cos x}{\sin x}$

$= +\cot x$ //

ii) $+\cot x = \frac{d}{dx}(\ln(\sin x))$

$\int \cot 3x \, dx$

$= \frac{1}{3}\ln(\sin 3x) + c$

~~cotx~~

b) $\int \frac{dx}{9+4x^2} = \int \frac{dx}{4\left(\frac{9}{4}+x^2\right)}$

$= \frac{1}{4}\int \frac{dx}{\frac{9}{4}+x^2}$

$= \frac{1}{4}\int \frac{dx}{\left(\frac{3}{2}\right)^2+(x)^2}$

$= \frac{1}{4}\left[\frac{1}{3/2}\tan^{-1}\frac{x}{3/2}\right]+C$ ✓(2)

$= \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$ //

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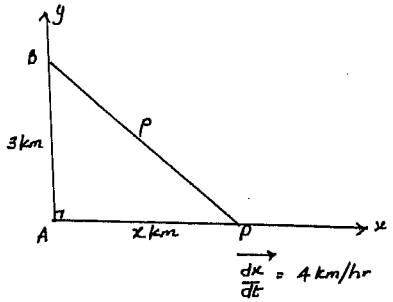
c) $\frac{dx}{dt} = 4$

want $\frac{dp}{dt}$ when $p = 4$

$\frac{dP}{dt} = \frac{dx}{dt} \cdot \frac{dP}{dx}$ ✓ (1)



$p^2 = x^2 + 3^2$

$\therefore p = \sqrt{x^2+9} = (x^2+9)^{1/2}$

$\frac{dp}{dx} = \frac{1}{2}(x^2+9)^{-1/2} \cdot 2x$

$= \frac{x}{\sqrt{x^2+9}}$

$\therefore \frac{dp}{dt} = \frac{dp}{dx} \times \frac{dx}{dt}$

$= \frac{x}{\sqrt{x^2+9}} \times 4$

$= \frac{4}{\sqrt{4^2+9}} \times 4$

$= \frac{16}{5}$ km/hr

$= 3\frac{1}{5}$ km/hr

Question 6

a) $y = \dfrac{e^{-x}}{1+x^2}$

$\dfrac{dy}{dx} = \dfrac{-e^{-x}(1+x^2) - 2xe^{-x}}{(1+x^2)^2}$ $u = e^{-x}$ $v = 1+x^2$, $u' = -e^{-x}$ $v' = 2x$

$= \dfrac{e^{-x}(-1-x^2-2x)}{(1+x^2)^2}$

$= \dfrac{-e^{-x}(x+1)^2}{(1+x^2)^2}$ ✓ 1

When $x=-1$, $\dfrac{dy}{dx}=0$

But for all other values of x

$(x+1)^2 > 0$

$-e^{-x}(x+1)^2 < 0$ since $e^{-x} > 0$

$\therefore$ decreasing for all x except $x=-1$.

and what does this mean?

b) i) $P = A(1-e^{-kt})$

$P = A - Ae^{-kt} \Rightarrow Ae^{-kt} = A-P$

$\dfrac{dP}{dt} = kAe^{-kt}$

$= k(A-P)$ as reqd. // ✓ 1

ii) For maximum, $\dfrac{dP}{dt} = 0$

$k(A-P)=0$

$kA - kP = 0$

$kA = kP$

$A = \dfrac{kP}{k}$ ✓

$\therefore P = A$ ✓ 1

$\therefore$ Max pop. is A //

iii) $\frac{1}{4}P = A(1-e^{-5k})$ ✓ 1

$e^{-5k} = \dfrac{3}{4}$

$-5k = \ln\left(\frac{3}{4}\right)$

$k = \dfrac{\ln\left(\frac{3}{4}\right)}{-5}$ ✓ 1

$\therefore$ when $t=10$

$P = A(1-e^{-10k})$ where $-10k = \dfrac{-10\ln 3/4}{-5}$

$= A(1-e^{\ln\frac{9}{16}})$ $= 2\ln\frac{3}{4}$

$= A(1-\frac{9}{16})$ $= \ln\frac{9}{16}$

$= \frac{7}{16}A$

$\frac{7}{16}$ is population.

no further work

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