Pymble ladies 2005 Ext 2 trial

Question 1 (15 marks) Use a separate writing booklet

MARKS

(a) Find $\int \tan 2x \sec 2x \, dx$. 1

(b) Find $\int \frac{1}{x} \sec^2(\ln x) \, dx$. 1

(c) Find $\int \frac{4x-x^2}{(x+1)(x^2+4)} \, dx$. 3

(d) Find $\int \cos 5x \sin 2x \, dx$. 2

(e) Evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x} \, dx$ using the substitution $t = \tan \frac{x}{2}$ 3

(f) Find $\int_0^{\frac{\pi}{2}} x \sin x \cos x \, dx$. 3

(g) Using the result $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$, show that

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx = \frac{\pi}{4}.$$ 2

Question 2 (14 marks) Use a separate writing booklet

MARKS

(a) Use the graph of $y = \ln x$ to sketch the graphs of:

(i) $y = \ln(-x)$ 1

(ii) $y = -\ln x$ 1

(iii) $y = |\ln x|$ 1

(b) Use the graphs of $y = x$ and $y = e^{-x}$ to sketch the graph of $y = xe^{-x}$. 2

(c) Use the graph of $y = x^2 - 1$ to sketch the graph of $y = (x^2-1)^2$. 2

(d) For the function $f(x) = 3x - \frac{x^3}{4}$, use the graph of $y = f(x)$ to sketch the graph of $y^2 = f(x)$. 2

(e) Use the graphs of $y = 2^u$ and $u = \cos x$ $(0 \le x \le 2\pi)$ to sketch the graph of $y = 2^{\cos x}$ $(0 \le x \le 2\pi)$. 2

(f) Sketch the graph of $y = \sin 2x$ for $0 \le x \le 2\pi$. Use this graph to solve the inequality $|\sin 2x| \ge \frac{1}{2}$, for $0 \le x \le 2\pi$. 3

Question 3 (15 marks) Use a separate writing booklet

MARKS

(a) Solve for z where $z \in \mathbb{C}$ — 2

$$z^2 + 2iz + 2 = 0.$$

(b) Form a quadratic equation whose roots are $4i$ and $3+i$. — 2

(c) If $w = 1+2i$ and $z = 2-3i$, express in the form $a+ib$.

(i) $w+z$ — 1

(ii) $w\overline{z}$ — 2

(iii) $\frac{w}{z}$ — 2

(d) Express $\sqrt{3}-i$ in the form $r(\cos\theta + i\sin\theta)$ and plot on the Argand diagram showing θ, r and the Cartesian coordinates. — 3

(e) By expanding $(\cos\theta + i\sin\theta)^4$, find expressions for $\cos 4\theta$ and $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$. Hence deduce an expression for $\tan 4\theta$ in terms of powers of $\tan\theta$. — 3

Question 4 (15 marks) Use a separate writing booklet

MARKS

(a) For the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ find:

(i) the eccentricity; — 1

(ii) the coordinates of the foci; — 1

(iii) the equations of the directrices. — 1

(iv) Sketch the ellipse showing essential features. — 1

(b) Find the equation of the tangent to the hyperbola $\frac{x^2}{12} - \frac{y^2}{27} = 1$ at the point (4, 3). — 2

(c) A point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. — 4

The line through P perpendicular to the x-axis meets an asymptote at Q and the normal at P meets the x-axis at N.
Show that QN is perpendicular to the asymptote.

(d) The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$. — 5

The normal at P meets the hyperbola again at Q.
M is the midpoint of PQ. Find the equation of the locus of M.

Question 5 (15 marks) Use a separate writing booklet

MARKS

(a) Find $P(x)$, given that $P(x)$ is monic, of degree 3, with 5 as a single zero and -2 as a zero of multiplicity 2. **2**

(b) Find the remainder when $P(x)=x^3+2x^2+1$ is divided by $x+i$. **1**

(c) If $P(x)=x^4-2x^3-x^2+6x-6$ has a zero $1-i$, find the zeros of $P(x)$ over $\mathbb{C}$, and factorise $P(x)$ fully over $\mathbb{R}$. **3**

(d) Solve the equation $18x^3+27x^2+x-4=0$, given the roots are in arithmetic progression. **3**

(e) The equation $x^3+x^2-2x-3=0$ has roots α, β and γ. Find the equations with roots:

(i) $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$; **1**

(ii) $\alpha+2$, $\beta+2$ and $\gamma+2$. **1**

(f) The equation $x^3+x^2+2=0$ has roots α, β and γ. Evaluate:

(i) $\alpha^3+\beta^3+\gamma^3$ **2**

(ii) $\alpha^4+\beta^4+\gamma^4$ **2**

Question 6 (15 marks) Use a separate writing booklet

MARKS

(a) If $I_n=\int_0^{\frac{\pi}{2}} x^n \cos x\, dx$ for $n\geq 0$, show that $I_n=\left(\frac{\pi}{2}\right)^n-n(n-1)I_{n-2}$ for $n\geq 2$. Hence evaluate I_6. **5**

(b) By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region, determined by $0\leq x\leq 2$ and $0\leq y\leq x^3$, about the line $y=8$. **5**

(c) (i) Let R be the region in the plane for which $0\leq x\leq\frac{\pi}{2}$ and $0\leq y\leq\sin x$. Sketch R. **1**

(ii) A solid is formed by rotating the region R about the y-axis. Use the method of cylindrical shells to find the volume of the solid. **4**

Question 7 (15 marks) Use a separate writing booklet

MARKS

(a) The rise and fall of the tide at Bedrock Harbour may be taken as simple harmonic, the interval between successive high tides being 12½ hours. The harbour entrance has a depth of 15m at high tide and 7m at low tide.
If low tide occurs at 11am on a certain day, find the earliest time thereafter that a ship requiring a minimum depth of 13m of water can pass through the entrance. **7**

(b) Use Mathematical Induction to prove DeMoivre's Theorem ie. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all positive integer values of n. **4**

(c) Let $f(x) = \begin{pmatrix} \dfrac{\sin x}{x} & \text{for} & 0 < x < \dfrac{\pi}{2} \\ 1 & \text{for} & x = 0 \end{pmatrix}$

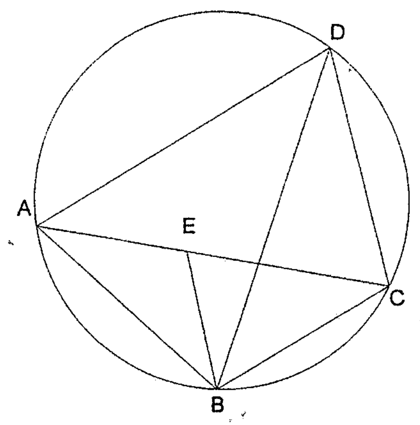
(i) Find the derivative of $f(x)$ for $0 < x < \frac{\pi}{2}$ and prove that $f'(x)$ is negative in this interval. **2**

(ii) Sketch the graph of $y = f(x)$ for $0 < x < \frac{\pi}{2}$ and deduce that $\sin x > \frac{2x}{\pi}$ in this interval. **2**

Question 8 (15 marks) Use a separate writing booklet

MARKS

(a)

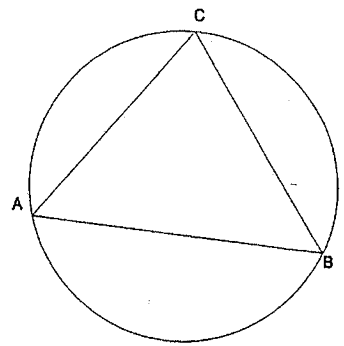


In the diagram $ABCD$ is a cyclic quadrilateral. E is a point on AC such that $\angle ABE = \angle DBC$.

(i) Show that $\Delta ABE \,|||\, \Delta DBC$ and $\Delta ABD \,|||\, \Delta EBC$. **2**

(ii) Hence show that $AB.DC + AD.BC = AC.DB$ **2**

(iii)



In the diagram ABC is an equilateral triangle inscribed in a circle.
P is a point on the minor arc AB of the circle.
Use the result in part (ii) to show that $PC = PA + PB$. **2**

Question 8 (continued)

MARKS

(b) (i) Prove that $a^2 + b^2 \geq 2ab$ where a, b are any two real numbers. **2**

(ii) If a, b and c are three real, positive numbers all less than 1, such that $a + b + c > abc$, prove that $a^2 + b^2 + c^2 > abc$. **2**

(c) When a particle is projected vertically upwards from the moon's surface, its distance x from the centre of the moon is given by

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -f\,\frac{R^2}{x^2}$$

where v is the upward speed, R is the radius of the moon and f is the acceleration due to gravity at the moon's surface and any possible atmospheric resistance is neglected. If v_0 is the speed of projection, show that:

(i) $v^2 = \dfrac{2fR^2}{x} + v_0^2 - 2fR$; **2**

(ii) the maximum height H, above the moon's surface, to which the particle will ascend is given by **2**

$$H = \frac{R v_0^2}{2fR - v_0^2}.$$

(iii) Taking $R \approx 1800\ km$, $f \approx 1.6\ ms^{-2}$, estimate the escape velocity of the particle from the moon in kms^{-1}. **1**

$(ab)^2 + (bc)^2 \geq 2ab^2c$

$(ac)^2 +$

$2[(ab)^2 + (bc)^2 + (ca)^2] \geq$

$abc[a + b + c]$

$(a-b)^2 = a^2 - 2ab + b^2$

$(a-b)^2 \geq 0$

$\therefore a^2 - 2ab + b^2 \geq 0$

$\therefore \boxed{a^2 + b^2 \geq 2ab}$

Similarly, $b^2 + c^2 \geq 2bc$

and, $a^2 + c^2 \geq 2ac$

$c^2 \geq 2ac - a^2$

R

$0 < a, b, c < 1$

$\boxed{a + b + c > abc}$

$\therefore 2(a^2 + b^2 + c^2) \geq 2(ab + bc + ac)$

$a^2 + b^2 + c^2 \geq ab + bc + ac$

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ac).$

$\therefore a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab + bc + ac)$

Rymble Ladies 2005 — Dong

Q1. (a). $\int \tan 2x \sec 2x \, dx$

$= \frac{1}{2}\sec 2x + C$ ✓

(b). $\int \frac{1}{x}\sec^2(\ln x)\, dx$

$= \tan(\ln x) + C$ ✓

(c). $\int \frac{4x - x^2}{(x+1)(x^2+4)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+4)}$

$4x - x^2 = A(x^2+4) + (Bx+C)(x+1)$

Let $x = -1$ — Compare x^2 — Compare constants

$-5 = +A(5)$ — $-1 = A + B$ — $0 = 4A + C$

$-1 = +A$ — $-1 = -1 + B$ — $= -4 + C$

$A = -1$ ✓ — $B = 0$ ✓ — $C = +4$ ✓

$I = \int \frac{-1}{(x+1)} + \frac{4}{x^2+4}$

$= -\ln(x+1) + 2\tan^{-1}(x/2) + C$ ✓

(d). $\int \cos 5x \sin 2x \, dx$

~~$u = \cos 5x$, $v = \frac{1}{2}\sin 2x$ ...~~

~~$I = [\ldots]$~~

NB: $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

$\int \cos 5x \sin 2x \, dx$

$\therefore \frac{A+B}{2} = 5x$ — $\frac{A-B}{2} = 2x$ ✓

$\therefore A = 7x,\ B = 3x$

$I = \int \sin 7x - \sin 3x$

$= \left[\frac{-\cos 7x}{7} + \frac{\cos 3x}{3}\right]$ ✓

(e). $\int_0^{\pi/3} \frac{1}{1-\sin x}\, dx$

Let $t = \tan x/2$

$\sin x = \frac{2t}{1+t^2}$ — At $x = \pi/3,\ t = \frac{1}{\sqrt{3}}$; $x = 0,\ t = 0$.

$\frac{2dt}{(t^2+1)} = dx$ ✓

$I = \int_0^{1/\sqrt{3}} \frac{1}{1 - \frac{2t}{1+t^2}} \cdot \frac{2dt}{(t^2+1)}$

$= 2\int_0^{1/\sqrt{3}} \frac{1}{1-2t+t^2}\, dt$ ✓

$= 2\int_0^{1/\sqrt{3}} \frac{dt}{(1-t)^2}$

$= 2\left[\frac{1}{(1-t)}\right]_0^{1/\sqrt{3}}$ ✓

$= 2\left[\left(\frac{1}{1-\frac{1}{\sqrt{3}}}\right) - 1\right]$

(f) $\int_0^{\pi/2} x \sin x \cos x \, dx$

$= I = \int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) \sin x \cos x \, dx$

$= \frac{\pi}{2}\int_0^{\pi/2} \sin x \cos x - \int_0^{\pi/2} x \sin x \cos x \, dx$

$2I = \pi/4 \int_0^{\pi/2} \sin 2x \, dx$ ✓

$= -\pi/8 \left[\cos 2x\right]_0^{\pi/2}$

$= -\frac{\pi}{8}\left[-1 - 1\right]$ ✓

$= \pi/4.$

$\therefore I = \pi/8.$

(15)

OR

$\int x \sin x \cos x$

$u = x$ — $v = -\frac{1}{4}\cos 2x$

$u' = 1$ — $v' = \frac{1}{2}\sin 2x$ ✓

$I = \left[x \cdot \frac{-1}{4}\cos 2x\right]_0^{\pi/2} + \frac{1}{4}\int \cos 2x$

$= \left[\frac{+\pi}{8}\right] + \frac{1}{8}\left[\sin 2x\right]_0^{\pi/2}$ ✓

$= \pi/8.$

(g). $\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}$

$I = \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}$ ✓

$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\cos x + \sin x}\, dx$

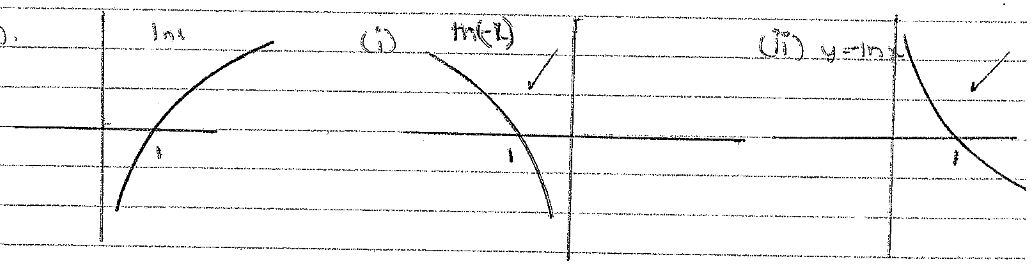
$= \int_0^{\pi/2} dx$

$= \left[x\right]_0^{\pi/2}$ ✓

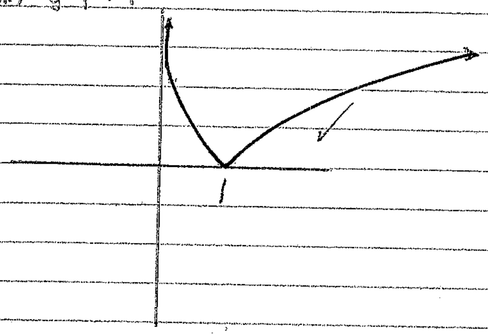
$2I = \pi/2$

$I = \pi/4$

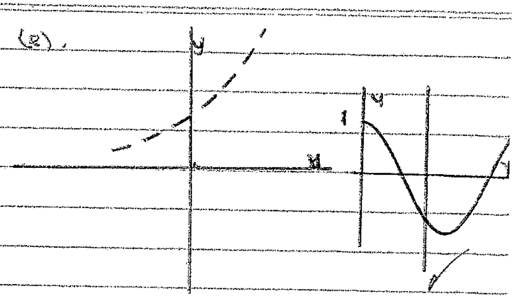
2. (a). (i) ✓ (ii) $y = -\ln x$ ✓



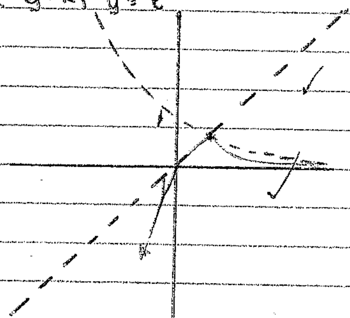
(a)(iii) $y = |\ln x|$



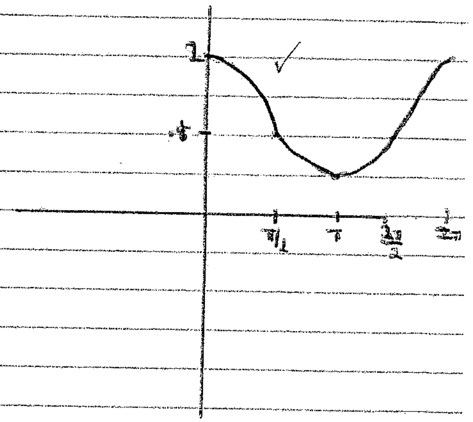
(e).



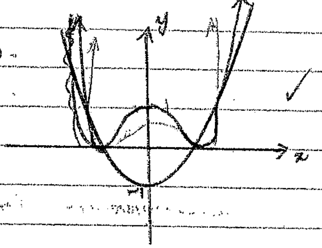
(b). $y = x$, $y = e^{-x}$



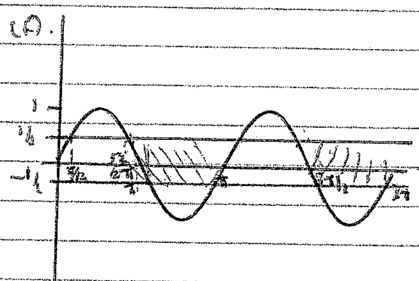
(12)



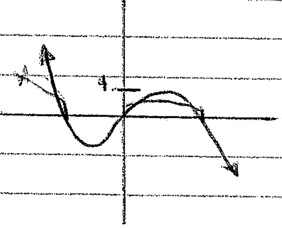
(c).



(f).



(d) $f(x) = 3x - \frac{x^3}{4}$. $\Rightarrow \frac{x}{4}(12 - x^2)$



$0 \le x \le \frac{\pi}{12} \cup \frac{\pi}{12} \le x \le \frac{7\pi}{12} \cup \frac{11\pi}{12} \le x \le \frac{13\pi}{12}$
$\cup \frac{17\pi}{12} \le x \le \frac{19\pi}{12} \cup \frac{23\pi}{12} \le x \le 2\pi$

Note: $|\sin 2x| \ge \frac{1}{2}$

$\Rightarrow -(\sin 2x) \ge \frac{1}{2}$ or $\sin 2x \ge \frac{1}{2}$

$\sin 2x \le -\frac{1}{2}$

$\frac{\pi}{12} \le x \le \frac{5\pi}{12}$

$\frac{7\pi}{12} \le x \le \frac{11\pi}{12}$ $\quad$ $\frac{13\pi}{12} \le x \le \frac{17\pi}{12}$

$\frac{19\pi}{12} \le x \le \frac{23\pi}{12}$

3.(a). $z^2 + 2iz + 2 = 0$.

$z = \frac{-2i \pm \sqrt{-4-8}}{2}$

$= \frac{-2i \pm \sqrt{12}i}{2}$.

$= -i \pm \sqrt{3}i$

(b). $\alpha = 4i$, $\beta = 3+i$

$\alpha + \beta = 3 + 5i$, $\alpha\beta = -4 + 12i$

$\therefore x^2 - (3+5i)x + (-4+12i)$

(14)

(c). $w = 1+2i$, $z = 2-3i$

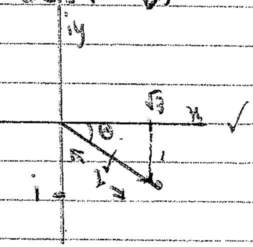
(i) $w + z = 3 - i$

(ii) $w\bar{z} = (1+2i)(2+3i)$

$= -4 + 7i$

(iii) $\frac{w}{z} = \frac{1+2i}{2+3i} \times \frac{2+3i}{2+3i} = \frac{-4+7i}{13}$.

(d). $\sqrt{3} - i = 2\operatorname{cis}(-\pi/6)$



(e). $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 1 - 8\cos^2\theta + 8\cos^4\theta$

$\sin 4\theta = -4\cos\theta\sin^3\theta + 4\cos^3\theta\sin\theta$

$= -4\cos\theta$

$\therefore \tan 4\theta = \dfrac{\dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta}}{\dfrac{\cos^4\theta(1 - 8\cos^2\theta + 8\cos^4\theta)}{\cos^4\theta}}$

$= \frac{4\tan\theta - 4\tan^3\theta}{\sec^4\theta - 6\sec^2\theta + 8}$.

$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(a) (i) $\frac{x^2}{16}+\frac{y^2}{25}=1$

$a=4, b=5$. $b>a$

$a^2=b^2(1-e^2)$ ✓

$16/25 = 1-e^2$

$e^2 = 1-\frac{16}{25}$ ✓

$e = 3/5$.

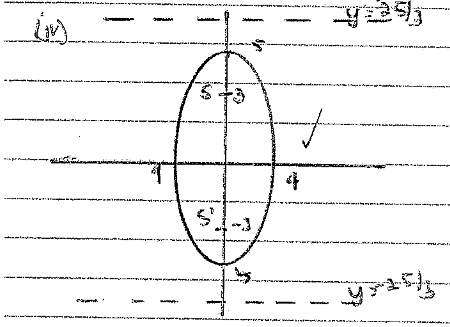
(ii) $\cancel{S(\pm ae, 0)}$ $S(0, \pm be)$

$= S(\ldots)$ $S = (0, 3)$ ✓

(iii) Directrix: $y = \pm b/e$

$= \pm 25/3$ ✓

(iv)



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(b) $\frac{x^2}{12}-\frac{y^2}{27}=1$

$f'(x)$: $\frac{2x}{12}-\frac{2y}{27}\frac{dy}{dx}=0$.

$\frac{x/6}{2y/27}=\frac{dy}{dx}$

$\frac{27x}{4\times 3y}=\frac{dy}{dx}$ ✓

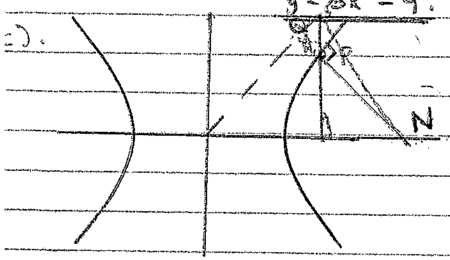
At $(4,3)$, $m = 3$

$y-3=3(x-4)$

$y-3=3x-12$ ✓

$y=3x-9$.

(c)



Normal. $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$

$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

N = when $y=0$.

$\frac{ax}{\sec\theta}=a^2+b^2$

$x = \sec\theta(a^2+b^2)$ /

$N: \left(\frac{\sec\theta(a^2+b^2)}{a}, 0\right)$

Q lies on $y = b/a\, x$ with x value $a\sec\theta$

$Q = (a\sec\theta, b\sec\theta)$ ✓

$m_{NQ} = \frac{b\sec\theta}{a\sec\theta - \frac{\sec\theta(a^2+b^2)}{a}}$

$= \frac{ab}{a^2-a^2-b^2}$

$= \frac{-ab}{b^2}$

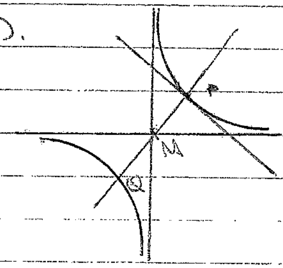
$= -a/b$ ✓

$m_{Asy} = b/a$.

$\therefore m_{Asy}\times m_{NQ} = -1$ ✓

$QN \perp Asy$

(d).



$xy = c^2$

$f'(x)$: $x\frac{dy}{dx}+y=0$.

$\frac{dy}{dx} = -y/x$

At $P(ct, \frac{c}{t})$

$m = -\frac{1}{t^2}$, $m_2 = t^2$

Eqn: $y-\frac{c}{t} = t^2(x-ct)$ ✓

$\Longrightarrow$ sub $xy=c^2$, $y=\frac{c^2}{x}$

$\frac{c^2}{x}-\frac{c}{t} = t^2x^2 - ct^2$

$c^2t = t^3x^2 + cx(1-t^4)$ ✗

$x^2t^3 + cx(1-t^4) - ct = 0$.

Roots, $\alpha+\beta$, $\therefore \Sigma_x$

$= \frac{-b}{2a}$, $x: \frac{-c(1-t^4)}{2t^3}$ ✓

Eqn, sub $x = \frac{c^2}{y}$ $\Longrightarrow$ $y-\frac{c}{t} = t^2\left(\frac{c^2}{y}-ct\right)$

$y - \frac{c}{t} = \frac{t^2c^2}{y} - ct^3$

$y^2 - \frac{cy(1-t^4)}{t} - t^3c^2 = 0$.

$\therefore \Sigma_y = \frac{c(1-t^4)}{2}$.

$x = \frac{-c(1-t^4)}{2t^3}$ $\quad y = \frac{+c(1-t^4)}{2t}$

$\therefore x = \frac{-2y}{2t^2}$ $\Longrightarrow$ $x = -y/t^2$

New $xy = \frac{-c^2(1-t^4)^2}{4t^4}$

But $t^2 = -\frac{y}{x}$

$\therefore xy = \frac{-c^2\left(1-\frac{y^2}{x^2}\right)^2}{4\left(\frac{y^2}{x^2}\right)}$ is ~~the~~ locus.

Q5.

(a). $P(x) = (x-5)(x+2)^2$

$= (x-5)(x^2+4x+4)$ ✓

$= x^3 + 4x^2 + 4x - 5x^2 - 20x - 20.$

$= x^3 - x^2 - 16x - 20.$ ✓

(b). $P(x) = x^3 + 2x^2 + 1$

$P(i) = (i)^3 + 2(i)^2 + 1$

$= -i - 2 + 1$

$= -1 - i$ ✓

(c). $P(x) = x^4 - 2x^3 - x^2 + 6x - 6$

$1-i$ is root, ∴ $1+i$ is a root

∴ $x^2 - 2x + 2$ is a factor.

$$\begin{array}{r|l} & x^2 - 3 \\ x^2-2x+2 & x^4 - 2x^3 - x^2 + 6x - 6 \\ & x^4 - 2x^3 + 2x^2 \\ & -3x^2 + 6x - 6 \\ & -3x^2 + 6x - 6 \\ & 0 \end{array}$$ ✓

∴ Roots: $1-i, 1+i, \sqrt{3}, -\sqrt{3}$ ✓

Over $\mathbb{R}$: $(x^2-2x+2)(x^2-3)$ ✓

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(d). $18x^3 + 27x^2 + x - 4 = 0.$

$\alpha - d, \alpha, \alpha + d$

$\sum\alpha = 3\alpha = -\frac{3}{2}$

$\alpha = -\frac{1}{2}.$ ✓

$\alpha\beta\gamma = \alpha(\alpha^2 - d^2) = 2/9.$

$-\frac{1}{2}(\frac{1}{4} - d^2) = 2/9$ ✓

$\frac{1}{4} - d^2 = -\frac{4}{9}$

$d^2 = 25/36$

$d = 5/6.$ ✓

Roots: $1/3, -1/2, -4/3$ ✓

(e)(i) $x^3 + x^2 - 2x - 3 = 0.$

$\alpha/2$ For, $y = x/2$ Let $x = 2y$

$(2y)^3 + (2y)^2 - 4y - 3 = 0.$

$8y^3 + 4y^2 - 4y - 3 = 0.$

∴ $8x^3 + 4x^2 - 4x - 3 = 0.$ ✓

(ii). For $\alpha + 2$

Let $y = x+2$, $y - 2 = x$

$(y-2)^3 + (y-2)^2 - 2(y-2) - 3 = 0.$

$(x-2)^3 + (x-2)^2 - 2x + 1 = 0.$

(f). $x^3 + x^2 + 2 = 0.$

For $\alpha^3 + \beta^3 + \gamma^3$, let $\alpha = x$ ∴ $\alpha^3 = -\alpha^2 - 2.$ ✓

∴ $\alpha^3 + \beta^3 + \gamma^3 = -(\alpha^2 + \beta^2 + \gamma^2) - 6.$

$= -[(\sum\alpha)^2 - 2\sum\alpha\beta] - 6.$

$= -(1 - 0) - 6.$ ✓

(f)(ii) For $\alpha^4 + \beta^4 + \gamma^4$

Let $\alpha = x$.

$\alpha^3 + \alpha^2 + 2 = 0.$

Times by α ✓

$\alpha^4 = -\alpha^3 - 2\alpha$

∴ $\alpha^4 + \beta^4 + \gamma^4 = -(\alpha^3 + \beta^3 + \gamma^3) - 2(\alpha + \beta + \gamma)$

$= -(-7) - 2(-1)$

$= 7 + 2.$ ✓

$= 9.$

6. (a) $I_n = \int_0^{\pi/2} x^n \cos x \, dx$

$u = x^n$ $v = \sin x$

$u' = nx^{n-1}$ $v' = \cos x$ ✓

$I_n = [x^n \sin x]_0^{\pi/2} - n\int x^{n-1} \sin x$

$= (\pi/2)^n - n\int x^{n-1} \sin x$

$u = x^{n-1}$ $v = -\cos x$

$u' = (n-1)x^{n-2}$ $v' = \sin x$

$I_n = (\pi/2)^n - n[x^{n-1}\cos x]_0^{\pi/2} - n(n-1)\int x^{n-2}\cos x \, dx$

$= (\pi/2)^n - n(n-1) I_{n-2}$

$I_6 = (\pi/2)^6 - 6(5) I_4$

$I_4 = (\pi/2)^4 - 4 \times 3 I_2$

$I_2 = (\pi/2)^2 - 2I_0$ ✓

$I_0 = \int_0^{\pi/2} \cos x \, dx$

$= [\sin x]_0^{\pi/2}$

$= 1$

∴ $I_2 = (\pi/2)^2 - 2.$

$I_4 = (\pi/2)^4 - 12(\pi/2)^2 + 2$ ✓

$I_6 = (\pi/2)^6 - 30[(\pi/2)^4 - 12(\pi/2)^2 + 2]$ ✓

$= (\pi/2)^6 - 30(\pi/2)^4 + 450(\pi/2)^2 - 60.$

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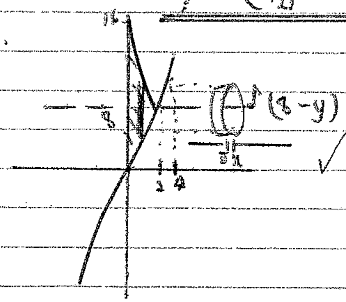
(b).



✓

$V = \pi A(x) . dx$ $y = x^3$

$= \pi\int_0^2 (8-y)^2 . dx$ ✓

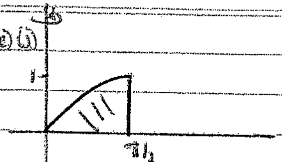
$= \pi\int_0^2 (8-x^3)^2 . dx$

$= \pi\int_0^2 64 - 2x^3 + x^6 . dx$ ✓

$= \pi\left[64x - \frac{2x^4}{2} + \frac{x^7}{7}\right]_0^2.$

$= \pi\left[\frac{968\pi}{7}\right]$

(6)(i)



(ii) $V = 2\pi \int r h \, dx$

$= 2\pi \int xy \, dx$

$= 2\pi \int_0^{\pi/2} x \sin x \, dx$

$u = x \qquad v = -\cos x$

$u' = 1 \qquad v' = \sin x$

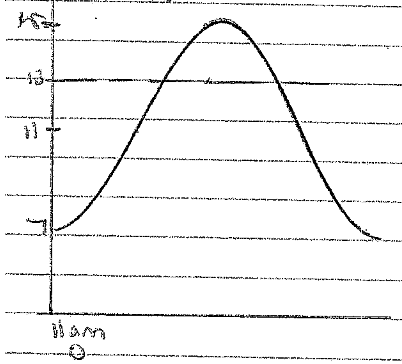
$V = 2\pi \left(\left[x \cos x \right]_0^{\pi/2} - \int \cos x \, dx \right)$

$= 2\pi \left[+ \sin x \right]_0^{\pi/2}$

$= 2\pi$

7. $T = \frac{2\pi}{n} = 12^{1/2}$.

$\frac{4\pi}{25} = n$



$y = 11 + 4 \cos\left(\frac{4\pi}{25} t\right)$

For 13 m

$13 = 11 + 4 \cos\left(\frac{4\pi}{25} t\right)$

$-1/_2 = \cos\left(\frac{4\pi}{25} t\right)$

$2\pi/_3 = \frac{4\pi}{25} t$.

$t = 25/6$

$\therefore$ Earliest : 3:10 pm

(b). $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Let $n = 1$,

LHS: $\cos\theta + i\sin\theta$ RHS: $\cos\theta + i\sin\theta$

True.

Assume for S(k)

Prove for $n = k+1$ RHS: $\cos(k+1)\theta + i\sin(k+1)\theta$

LHS $(\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k (\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$= \cos k\theta \cos\theta + i\sin\theta \cos k\theta + i\sin k\theta \cos\theta - \sin k\theta \sin\theta$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

$=$ RHS.

If true for $n = k$, then by the principles of Mathematical Induction, true for $n = k+1$ and all positive integer values of n.

$f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } (0 < x \le \pi/2) \\ 1 & \text{for } x = 0 \end{cases}$

$f'(x)$ $\quad u = \sin x \qquad v = x$

$u' = \cos x \qquad v' = 1$

$f'(x) = \frac{x \cos x - \sin x}{x^2}$ $\Rightarrow$ since $\tan x > x$ for $0 < x < \frac{\pi}{2}$

$\therefore \frac{\sin x}{\cos x} > x$

T.P at $f'(x) = 0$. $x \cos x = \sin x$

$x \tan x = 1$

$\therefore x \ne 0$.

$\sin x > x \cos x$ since $\cos x > 0$

No T.P.

$\therefore \frac{x \cos x - \sin x}{x^2} < \frac{0}{x^2}$

$\therefore$ at $x = 0$; $f'(x) \ne 0$.

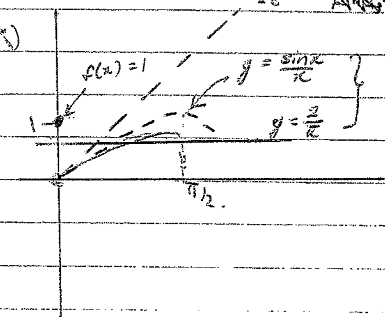
$x = \pi/_2$, $-$ve.

< 0.

Always

$\therefore$ Always Negative) $(0 < x < \pi/_2)$

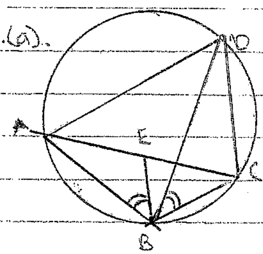
(ii)



$\frac{\sin x}{x} > \frac{2}{\pi}$

$\sin x > \frac{2x}{\pi}$ By the graph $\sin x > \frac{2x}{\pi}$

8.(a).



(i). In Δ's ABE & ADBC

$D\hat{B}C = A\hat{B}E$ (given)

$B\hat{D}C = B\hat{A}C$ (angles subtended on the same arc)

~~$\Delta ABE ||| DBC$~~

In Δ's ABD & EBC

$E\hat{B}C = A\hat{B}D$ (since $D\hat{B}C = A\hat{B}E$ + common)

$B\hat{A}C = B\hat{D}C$

$\therefore$ ABD ||| EBC

(ii) $\frac{AB}{DB} = \frac{AE}{DC}$ $\qquad \frac{AD}{BD} = \frac{EC}{BC}$

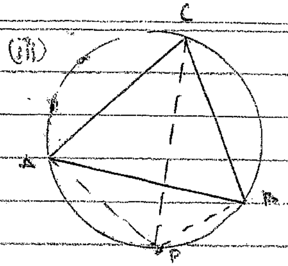
$AB \cdot DC = AE \cdot DB$ — ① $\qquad AD \cdot BC = BD \cdot EC$ — ②.

① + ② $\quad AB \cdot DC + AD \cdot BC = AE \cdot DB + BD \cdot EC$

$= DB(AE + EC)$

$AB \cdot DC + AD \cdot BC = DB \cdot AC$

(iii)



Δ $AP.BC + AC.PB = PC.AB$

But $AB = BC = CA$

$\therefore\ PC = PA + PB.$

$\therefore\ AP.\cancel{AC} + \cancel{AC}.PB = \cancel{AC}.PC$

$\therefore\ AP + PB = PC$ ... as req^d.

b). (i) $(a - b)^2 \geq 0.$

$a^2 - 2ab + b^2 \geq 0.$ ✓

$a^2 + b^2 \geq 2ab.$

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(ii) Similarly $a^2 + c^2 \geq 2ac$

$b^2 + c^2 \geq 2bc$

$\therefore\ 2(a^2 + b^2 + c^2) \geq 2(ab + ac + bc)$

$a^2 + b^2 + c^2 \geq ab + bc + ca \geq abbcca$ ✓

$\geq a^2b^2c^2$

$a^2 + b^2 + c^2 \geq abc.$ ✓

(c) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -f\frac{R^2}{x^2}$

$\frac{1}{2}v^2 = +f\frac{R^2}{x} + c.$

At $x = R$, $v = v_0$ ✓

$\therefore\ v_0^2 = 2f\frac{R^2}{R} + c$

$c = v_0^2 - 2f\frac{R^2}{R}$ ✓

$\therefore\ v^2 = 2f\frac{R^2}{x} + v_0^2 - 2fR$

(ii). Max Height $x = H$, $v = 0$.

$0 = 2f\frac{R^2}{H} + v_0^2 - 2fR$

$2fR - v_0^2 = \frac{2fR^2}{H}$

$H = \frac{2fR^2}{2fR - v_0^2}$ ✓

From v^2, As $x \to \infty$ $\frac{2fR^2}{x} \to 0.$

$v_0^2 - 2fR = 0.$

$2fR = v_0^2$

$\therefore\ H = \frac{Rv_0^2}{2fR - v_0^2}$ ✓

(iii) As $x \to \infty$ $\frac{2fR^2}{x} \to 0$

$\therefore\ 2fR = v_0^2$ ✓

If $R = 1800$, $f = 1.6$

$v_0^2 = 5760$

$v_0 \approx 75.89\ kms^{-1}$ ✓