Name: ______________________

Teacher's Name: ______________________

Mr Keanan-Brown
Mrs Hickey
Mrs Williams
Mrs Stock

PYMBLE LADIES' COLLEGE

2000 TRIAL H.S.C. EXAMINATION

MATHEMATICS

3/4 UNIT

Time Allowed: 2 hours
plus 5 minutes reading time

INSTRUCTIONS TO CANDIDATES:

1. All questions must be attempted.
2. All necessary working must be shown.
3. Start each question on a new page.
4. Put your name and your teachers' name on every sheet of paper.
5. Marks may be deducted for careless or untidy work.
6. Only approved calculators may be used.
7. DO NOT staple different questions together.
8. Hand this question paper in with your answers.
9. All rough working paper must be attached to the back of the last question.
10. All questions are of equal value.

There are seven (7) questions in this paper.

Stephanie Sun

Question 1 — **Marks**

(a) Find $\frac{d}{dx}(\sec 2x)$ — 1

(b) If $\log_m a = 0{\cdot}7$, $\log_m b = 0{\cdot}3$, $\log_m c = 0{\cdot}2$,
find the value of $\log_m \frac{\sqrt{a}}{b^2c^3}$ — 2

(c) Find the exact value of $\int_e^{e^2} \frac{dx}{x\ln x}$
(you may use the substitution $u = \ln x$ if you wish). — 3

(d) Evaluate $\lim_{x \to 0} \frac{\sin 2x}{5x}$ — 1

(e) Find $\int \sin x \cos x \, dx$ — 2

(f) (i) Sketch on the same diagram, $y = \frac{1}{x}$ and $y = \sqrt{x}$ — 3

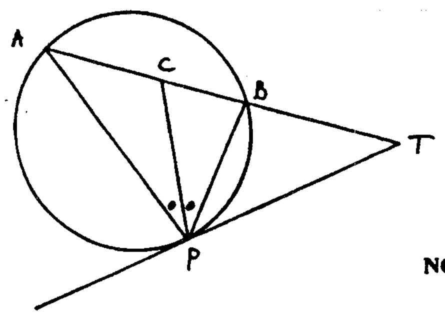
(ii) Hence, or otherwise, solve $\frac{1}{x} \geq \sqrt{x}$

Question 2 **(Start a new page)** **Marks**

(a) Use the substitution $u = \sqrt{x}$ to evaluate $\displaystyle\int_0^3 \frac{dx}{\sqrt{x}(x+1)}$ **4**

(b) Evaluate $\displaystyle\int_0^{\frac{1}{6}} \frac{3dx}{\sqrt{1-9x^2}}$ **3**

(c) **5**



NOT TO SCALE

A chord AB of a circle is produced to a point T. From T, a tangent is drawn, touching the circle at P. C is a point on AB such that CP bisects $\angle APB$.

(i) Copy the diagram onto your writing paper

(ii) Prove that $TP = TC$, giving reasons.

(iii) If $AT = 9$ and $TB = 4$, find TP and hence AC.

Units are in centimetres

Question 3 **(Start a new page)** **Marks**

(a) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \sin^2 2x\, dx$ **4**

(b) The area of a circle is A cm² and the circumference is C cm at time t seconds. **4**

If the area is increasing at a rate of 4 cm²/s, find the rate at which the circumference is increasing when the radius is 2 cm.

(c) (i) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$ where α is in radians. **4**

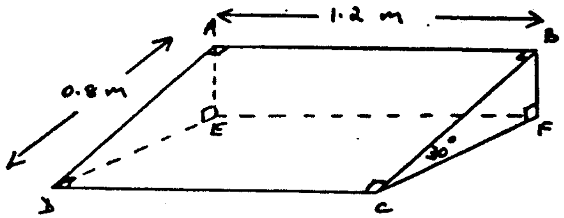
(ii) Hence, or otherwise, find the general solution of the equation $\sqrt{3}\cos\theta - \sin\theta = 1$

Question 4 **(Start a new page)** **Marks**

(a) Sketch $y = \tan^{-1} x$ 3

What is the maximum value that the gradient of the inverse tangent curve can have? Give reasons for your answer.

(b) 3



NOT TO SCALE

An architect's desk has a sloping work surface which measures 1.2 metres by 0.8 metres, as shown. The sloping work surface $ABCD$ makes an angle of 30° with the horizontal $EFCD$.

Find (i) the length of BF

(ii) the length of AC, correct to 2 decimal places

(iii) the angle that the diagonal AC makes with the horizontal, giving your answer to the nearest degree.

(c) The tangent at $P(6t, 3t^2)$ on the parabola $x^2 = 12y$ cuts the x, y axes at A, B respectively. O is the origin and C is the point such that $OACB$ is a rectangle. 6

Find (i) the equation of the tangent at P

(ii) the coordinates of A, B and C

(iii) the locus of C as P moves on the parabola.

Question 5 **(Start a new page)** **Marks**

(a) The velocity v ms^{-1} of a particle moving in simple harmonic motion along the x axis is given by $v^2 = 6 + 4x - 2x^2$ 8

(i) Between which two points is the particle oscillating?

(ii) What is the amplitude of the motion?

(iii) Find the acceleration of the particle in terms of x.

(iv) Write down the period of the oscillation.

(v) What is the maximum speed of the particle?

(b) Prove, by mathematical induction, that

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2}).....(1-\frac{1}{n^2}) = \frac{n+1}{2n} \text{ for all } n \geq 2.$$ 4

Question 6 **(Start a new page)** **Marks**

(a) The roots of the equation $x^3 - 6x^2 + 3x + k = 0$ are consecutive terms of an arithmetic sequence. Find the value of k. **5**

(b) Consider the function $f(x) = \dfrac{x-4}{x-2}$ for $x > 2$ **7**

(i) Show that $f(x)$ is an increasing function for all values of x in its domain.

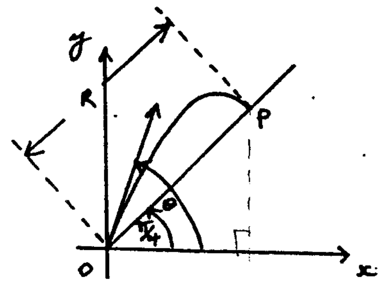
(ii) Explain briefly why the inverse function $f^{-1}(x)$ exists.

(iii) State the domain and range of $f^{-1}(x)$

(iv) Find the gradient of the tangent to $y = f^{-1}(x)$ at the point $(0, 4)$ on it.

Question 7 **(Start a new page)** **Marks**

12



A cat can jump with a velocity of $5ms^{-1}$. It is standing at O, at the bottom of a slope inclined at $\dfrac{\pi}{4}$ to the horizontal.

The cat jumps at an angle of θ to the horizontal, where $\dfrac{\pi}{4} < \theta < \dfrac{\pi}{2}$.

The equations of motion of the cat are $\ddot{x} = 0, \ddot{y} = -10$

(i) Use calculus to show that the coordinates of the cat's position at time t seconds are given by $x = 5t\cos\theta$ and $y = -5t^2 + 5t\sin\theta$.

(ii) The cat lands at P, where the length of $OP = R$ metres. Explain why $x = y = \dfrac{R}{\sqrt{2}}$ at P.

(iii) Show that $R = 5\sqrt{2}(\cos\theta\sin\theta - \cos^2\theta)$

(iv) By differentiation, find the value of θ for the cat to achieve maximum distance R.

(v) The cat had seen a mouse sitting 1.8m up the slope from 0. If the cat attains maximum distance R, will it need to run up the slope or down the slope in its attempt to catch the mouse (assuming the mouse remains stationary)? Justify your answer.

[Note: No animal was harmed in the writing of this question - the mouse escaped].

Stephanie Su

Pymble Ladies' College 2000 Trials 3/4 U

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Excellent work!

1) A.) $\frac{d}{dx}(\sec 2x) = \frac{d}{dx}(\cos 2x)^{-1}$

$= \frac{-1}{\cos^2 2x} \times -2\sin 2x = \frac{2\sin 2x}{\cos^2 2x}$

$= 2\tan 2x \sec 2x$ ✓

(Could have derived this from standard integrals for 1a.)

B) $\log_m \frac{\sqrt{a}}{b^2c^3}$

$= \log_m \sqrt{a} - \log_m b^2c^3$ ✓

$= \frac{1}{2}\log_m a - (2\log_m b + 3\log_m c)$

$= \frac{1}{2}(0.7) - 2(0.3) - 3(0.2) = \underline{-0.85}$ ✓

c) $\int_e^{e^2} \frac{dx}{x\ln x}$ $\quad u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$; $du = \frac{dx}{x}$ ✓

12

$= \int_1^2 \frac{du}{u}$ ✓

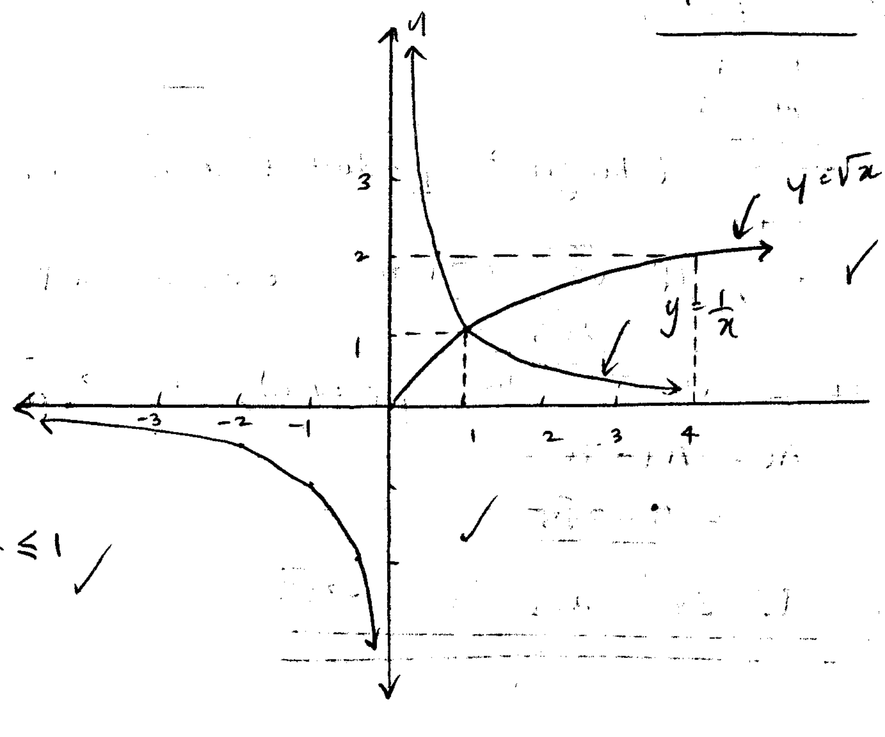
$= (\ln u)_1^2 = \ln 2 - \ln 1 = \underline{\ln 2}$ ✓

D) $\lim_{x\to 0} \frac{\sin 2x}{5x} = \lim_{x\to 0} \frac{\sin 2x}{2x} \times \frac{2}{5} = 1 \times \frac{2}{5} = \frac{2}{5}$ ✓

E) $\int \sin x\cos x\, dx = \frac{1}{2}\int \sin 2x\, dx = \frac{1}{2} \times -\frac{1}{2}\cos 2x + c$ ✓

$= \underline{-\frac{1}{4}\cos 2x + c}$ ✓

f) i.)



✓ ✓

ii.) $0 < x \leq 1$ ✓

Question 2

A.) $\displaystyle\int_0^3 \frac{dx}{\sqrt{x}(x+1)}$ $\qquad u = \sqrt{x}$; $x = u^2$

$\displaystyle \frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$\displaystyle = \int_0^{\sqrt{3}} \frac{2du}{u^2+1}$ $\qquad \displaystyle du = \frac{dx}{2\sqrt{x}}$

$= 2\left(\tan^{-1} u\right)_0^{\sqrt{3}} = 2\left(\tan^{-1}\sqrt{3}\right)$

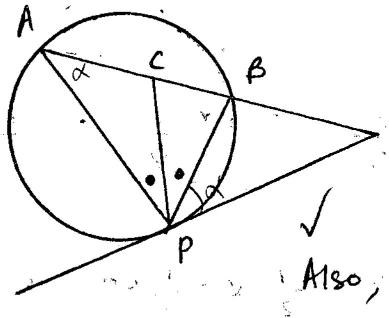
$\displaystyle = 2 \cdot \frac{\pi}{3} = \boxed{\frac{2\pi}{3}}$

B) $\displaystyle\int_0^{\frac{1}{6}} \frac{3dx}{\sqrt{1-9x^2}} = \int_0^{\frac{1}{6}} \frac{3dx}{\sqrt{9(\frac{1}{9}-x^2)}} = \int_0^{\frac{1}{6}} \frac{dx}{\sqrt{\frac{1}{9}-x^2}}$

$= \left(\sin^{-1} 3x\right)_0^{\frac{1}{6}}$

$\displaystyle = \sin^{-1}\frac{1}{2} = \boxed{\frac{\pi}{6}}$

c) i.)



ii.) Let $\angle BPT = \alpha$.

$\angle BPT = \angle PAB = \alpha$ ($\angle$ made by tgt and chord equals $\angle$ in alt. segment)

$\angle TCP = \alpha + \bullet$ (exterior $\angle$ equals sum of interior opp $\angle$)

Also, $\angle CPT = \alpha + \bullet$

$\therefore \angle CPT = \angle TCP$, $\therefore \Delta TCP$ is isos Δ (base $\angle$ of isos Δ equal)

$\therefore TP = TC$ (2 sides of isos ΔTCP equal)

11

iii.) Given: $AT = 9$

$TB = 4$

$\therefore AB = 5$

(Use similar Δ's to check theorem).

$TP^2 = \cancel{AB \times BT}$ $TA \times TB$ (tangent^2 = product of secant intercepts)

$TP^2 = \cancel{5 \times 4}$ 9×4

$TP^2 = \cancel{20}$ 36 ; $TP = \cancel{\sqrt{20}}$ 6 ($TP > 0$ because it is a positive length)

$= 2\sqrt{5}$

Since ~~TP = 2~~ $TP = TC$ (already proven), $TC = \cancel{2\sqrt{5}}$ 6

$\therefore AC = AT - \cancel{TC}$ 6

$= 9 - \cancel{2\sqrt{5}}$ 6

$\therefore TP = \cancel{2\sqrt{5}}$ 6 and $AC = \cancel{9-2\sqrt{5}}$ 3

Question 3

A) $\int_0^{\frac{\pi}{6}} \sin^2 2x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{6}} 1 - \cos 4x \, dx$ ✓

$= \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right)_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3}\right)$ ✓

$= \frac{1}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)$ ✓

$= \frac{\pi}{12} - \frac{\sqrt{3}}{16} = \boxed{\frac{4\pi - 3\sqrt{3}}{48}}$ ✓

B) $\frac{dA}{dt} = 4cm^2/s$

Find $\frac{dC}{dt}$ when $r = 2$.

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$ ✓

$C = 2\pi r$

$\frac{dC}{dr} = 2\pi$ ✓

Now $\frac{dA}{dt} \times \frac{dr}{dA} = 4 \times \frac{1}{2\pi r} = \frac{2}{\pi r}$. $\therefore \frac{dr}{dt} = \frac{2}{\pi r}$ ✓

$\frac{dC}{dt} = \frac{dC}{dr} \times \frac{dr}{dt}$

$= 2\cancel{\pi} \times \frac{2}{\cancel{\pi} r} = \frac{4}{r} \quad (r = 2)$

$= \frac{4}{2} = 2$

$\therefore \frac{dC}{dt} = 2cm/s$ ✓

c) i.) $\sqrt{3}\cos\theta - \sin\theta = R\cos(\theta+\alpha)$
$= R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$

$\sqrt{3} = R\cos\alpha$ — ①
$1 = R\sin\alpha$ — ②

$\frac{②}{①} = \tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}$ ✓

$R = \sqrt{a^2+b^2} = \sqrt{3+1} = 2$

$\therefore \sqrt{3}\cos\theta - \sin\theta = 2\cos(\theta + \frac{\pi}{6})$ ✓

ii.) $2\cos(\theta + \frac{\pi}{6}) = 1$

12

$\cos(\theta + \frac{\pi}{6}) = \frac{1}{2}$

$\theta + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$

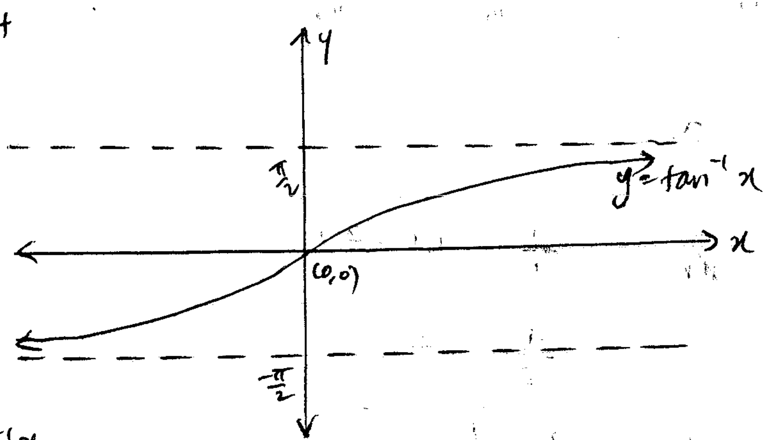
$\theta = 2n\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$. ✓

$\therefore \theta = 2n\pi + \frac{\pi}{3} - \frac{\pi}{6}$ OR $\theta = 2n\pi - \frac{\pi}{3} - \frac{\pi}{6}$

$\theta = 2n\pi + \frac{\pi}{6}$ OR $\theta = 2n\pi - \frac{\pi}{2}$ ✓

Question 4

a.)



✓

$y = \tan^{-1} x$

$\frac{dy}{dx} = \frac{1}{1+x^2}$

The max value of $\frac{dy}{dx}$ is when $x = 0$. because as $x \to \infty$, $\frac{dy}{dx} \to 0$

$\frac{dy}{dx} = \frac{1}{1+0} = 1$ $\therefore$ greatest value for gradient is $\underline{\underline{1}}$ ✓

b) i.) In ΔBCF, $\sin 30° = \frac{BF}{0.8}$

$BF = 0.8 \times \frac{1}{2} = \underline{0.4m}$ ✓

ii.) Using pythag. theorem,

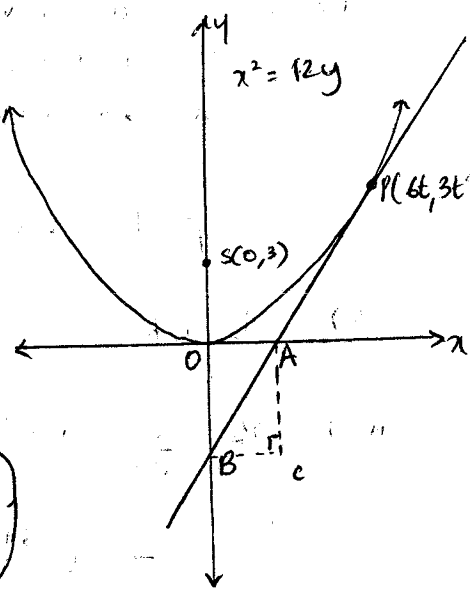
$1.2^2 + 0.8^2 = AC^2$

$AC = \sqrt{2.08} = \underline{1.44m\ (2dp)}$ ✓

iii) In ΔAEC, ~~sin~~ $\angle ACE = \frac{AE}{AC}$

$\sin \angle ACE = \frac{0.4}{\sqrt{2.08}}$

$\underline{\angle ACE = 16°61'}$ ✓



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c) $x^2 = 12y$

$4a = 12$; $a = 3$.

i.) $y = \frac{x^2}{12}$; $\frac{dy}{dx} = \frac{x}{6}$.

At P, grad. of tgt $= \frac{6t}{6} = t$. ✓

Eqt of tgt at P is :

$(y - 3t^2) = t(x - 6t)$

$y = tx - 6t^2 + 3t^2$

$\underline{y = tx - 3t^2}$ ✓

ii.) At A, $y = 0$

$tx - 3t^2 = 0$; $tx = 3t^2$

$x = 3t$

$\therefore \underline{A = (3t, 0)}$

At B, $x = 0$.

$y = 0 - 3t^2$ $\therefore \underline{B = (0, -3t^2)}$

C lies directly below A and to the right of B. ~~rectangle~~ OACB is a rectangle)

$\underline{C = (3t, -3t^2)}$ ✓

iii.) <u>Locus of C</u>

$x = 3t$ $\qquad$ $y = -3t^2$

$t = \frac{x}{3}$ $\longrightarrow$ $y = -3\left(\frac{x^2}{9}\right)$ ✓

$y = -\frac{x^2}{3}$

$-x^2 = 3y$ ✓

$\therefore \underline{x^2 = -3y}$ is locus of C

<u>Question 5</u>

A) $v^2 = 6 + 4x - 2x^2$

i.) Find x when $v = 0$.

$2x^2 - 4x - 6 = 0$ ✓

$x^2 - 2x - 3 = 0$; $(x-3)(x+1) = 0$

$\underline{x = 3, \quad x = -1}$

$\therefore$ particle is oscillating between $x = 3$ and $x = -1$

ii.) Amplitude $= \frac{3+1}{2} = \underline{2m}$ ✓

iii.) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(3 + 2x - x^2\right)$ ✓

$= 2 - 2x$

$\therefore \ddot{x} = 2 - 2x = \underline{-2(x-1)}$ ✓

iv.) Period $= \frac{2\pi}{n} = \frac{2\pi}{\sqrt{2}} = \underline{\sqrt{2}\pi \text{ sec}}$ ✓

v.) Max speed occurs when $\ddot{x} = 0$.

$-2(x-1) = 0$; $x - 1 = 0$; $\underline{x = 1}$

when $x = 1$, $v^2 = 6 + 4 - 2$ ✓

$= 8$

$v = \pm\sqrt{8}$ (speed $= |v|$)

$\therefore$ max. speed $= 2\sqrt{2}$ m/s ✓

B) Step 1

Let $n=2$.

LHS $1-\frac{1}{n^2} = 1-\frac{1}{4} = \frac{3}{4}$

RHS $\frac{n+1}{2n} = \frac{2+1}{4} = \frac{3}{4} = LHS.$

$\therefore$ true for $n=1$ ✓

12

Step 2

Assume true for $n=k$,

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$$

R.T.P. also true for $n=k+1$

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) = \frac{k+2}{2(k+1)}$$ ✓

LHS $$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right) = \underbrace{\frac{k+1}{2k}}_{\text{from assumption}}\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \frac{\cancel{k+1}}{2k}\left(\frac{(k+1)^2-1}{(k+1)^{2}}\right) = \frac{k^2+2k}{2k(k+1)}$$

$$= \frac{\cancel{k}(k+2)}{2\cancel{k}(k+1)} = \frac{k+2}{2(k+1)} = RHS$$ ✓

Step 3

If $n=k$ is true and $n=k+1$ is true and also $n=2$ is true, then $n=2+1=3$, $n=3+1=4$ and so on are true. $\therefore$ by the Principle of Mathematical Induction, it is true for all $n \geq 2$.

Question 6

A) Let $P(x) = x^3-6x^2+3x+k=0$

Let roots be α, $\alpha+d$ and $\alpha+2d$ $\Rightarrow$ Note: $\alpha-d, \alpha, \alpha+d$ are better choices.

Sum of roots $= 3\alpha = 6$

$\therefore \alpha = 2$, sub into

$P(2) = 8-24+6+k =$

$\therefore \underline{k=10}$.

> $\alpha+\alpha+d+\alpha+2d = 6$

$3\alpha+3d=6$; $\alpha+d=2$; $\alpha = 2-d$ — (1)

> $\alpha(\alpha+d)(\alpha+2d) = -k$ (sub in (1))

$(2-d)(d+2-d)(2-d+2d) = -k$ ✓

$2(2-d)(2+d) = -k$; $2(4-d^2) = -k$ — (2)

$> \alpha(\alpha+d)+\alpha(\alpha+2d)+(\alpha+d)(\alpha+2d)=3$

$(2-d)(2-d+d)+(2-d)(2-d+2d)+(2-d+d)(2-d+2d)=3$

$2(2-d)+(2-d)(2+d)+2(2+d)=3$

$\cancel{4}-\cancel{2d}+4-d^2+\cancel{4}4+\cancel{2d}=3$

$12-d^2=3$

$d^2=9\ ;\ \underline{d=\pm 3}$ (sub into (2))

$2(4-9)=-k$

$-10=-k\ ;\ \boxed{k=10}$

B) $f(x)=\dfrac{x-4}{x-2}$ for $x>2$ $\Rightarrow f(x)=\dfrac{x-2-2}{x-2}$

$=\dfrac{x-2}{x-2}-\dfrac{2}{x-2}=1-\dfrac{2}{x-2}$ $y\neq 1$

i.) $f'(x)=\dfrac{(x-2)-(x-4)}{(x-2)^2}=\dfrac{x-2-x+4}{(x-2)^2}=\dfrac{2}{(x-2)^2}$ which is >0 for all values of x

$\therefore f(x)$ is an increasing function for all values of x in its domain

ii.) An inverse function, $f^{-1}(x)$ exists because when a horizontal test is applied to $f(x)$, it only only cuts the graph at one spot, ie there is only 1 y-value for every x-value.

iii.) Range of $f^{-1}(x)$: $y>2$

Domain: ~~$x>1$~~ $x<1$ $x\neq 1$.

11

iv.) $y=f(x)=\dfrac{x-4}{x-2}$ (swap x and y-values)

$x=\dfrac{y-4}{y-2}$; $x(y-2)=y-4$

$xy-y=2x-4$

$y=\dfrac{2x-4}{x-1}$ $\therefore f^{-1}(x)=\dfrac{2x-4}{x-1}$

$=\dfrac{2(x-1)-2}{x-1}$

$=2-\dfrac{2}{x-1}$

$\dfrac{d}{dx}f^{-1}(x)=\dfrac{d}{dx}\left(2-\dfrac{2}{x-1}\right)=-2\left(\dfrac{-1}{(x-1)^2}\right)$

$=\dfrac{2}{(x-1)^2}$. When $x=0$, $\dfrac{d}{dx}f^{-1}(x)=\dfrac{2}{1}=2$

$\therefore$ grad. $=\underline{\underline{2}}$

Question 7

i.) $\ddot{x} = 0$

$\dot{x} = \text{constant}$. when $t=0$, $\dot{x} = V\cos\alpha$

$c = V\cos\alpha$

$\therefore \dot{x} = V\cos\alpha$. $(v=5, \alpha=\theta)$

$\therefore \dot{x} = 5\cos\theta$ ✓

$x = 5t\cos\theta + c$

when $t=0$, $x=0$

$0 = 0 + c$; $c=0$

$\therefore x = 5t\cos\theta$ ✓

$\ddot{y} = -10$

$\dot{y} = -10t + c$

when $t=0$, $\dot{y} = V\sin\alpha$

$V\sin\alpha = 0 + c$; $c = V\sin\alpha$

$\dot{y} = -10t + V\sin\alpha$ $(v=5, \alpha=\theta)$

$\therefore \dot{y} = -10t + 5\sin\theta$ ✓

$y = -5t^2 + 5t\sin\theta + c$.

when $t=0$, $y=0$

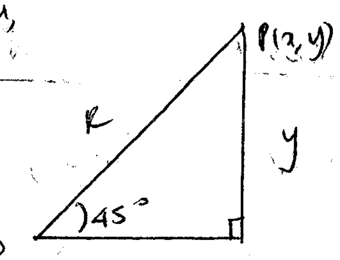
$0 = 0 + 0 + c$; $c=0$ ✓

$\therefore y = -5t^2 + 5t\sin\theta$

ii.) Let coord. of P be (x,y)

Using pythag. theorem,

$x^2 + y^2 = R^2$.



BUT we are given that PO makes an angle of 45° with horizontal.

$\therefore \tan 45 = \frac{y}{x}$

$1 = \frac{y}{x}$ $\therefore y = x$ ✓

$\therefore x^2 + x^2 = R^2$

$2x^2 = R^2$

$x^2 = \frac{R^2}{2}$

$x = \sqrt{\frac{R^2}{2}}$ ($x>0$ since it is a length)

$= \frac{R}{\sqrt{2}}$ ✓

$\therefore x = y = \frac{R}{\sqrt{2}}$

iii.) $x = 5t\cos\theta$; $t = \frac{x}{5\cos\theta}$ (sub into y eqt)

$y = -5\left(\frac{x^2}{25\cos^2\theta}\right) + 5\sin\theta\left(\frac{x}{5\cos\theta}\right) = -\frac{x^2}{5\cos^2\theta} + x\tan\theta = \frac{R}{\sqrt{2}}$

But $x = \frac{R}{\sqrt{2}}$.

$$\frac{-\frac{R^2}{2}}{5\cos^2\theta} + \frac{\cancel{R}}{\sqrt{2}}\tan\theta = \frac{\cancel{R}}{\sqrt{2}}$$ ✓

$$\frac{-R}{10\cos^2\theta} + \frac{\tan\theta}{\sqrt{2}} = \frac{1}{\sqrt{2}},$$

$$\frac{R}{10\cos^2\theta} = \frac{\tan\theta - 1}{\sqrt{2}}$$

$$R = \frac{10\cos^2\theta\,(\tan\theta - 1)}{\sqrt{2}}$$ ✓

$$R = \frac{10\sin\theta\cos\theta - 10\cos^2\theta}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$R = \frac{10\sqrt{2}(\sin\theta\cos\theta - \cos^2\theta)}{2} = \underline{5\sqrt{2}(\sin\theta\cos\theta - \cos^2\theta)}$$ ✓

iv.) $$\frac{dR}{d\theta} = 5\sqrt{2}\left(\cos 2\theta - 2\cos\theta . -\sin\theta\right)$$

$$= 5\sqrt{2}\,(\cos 2\theta + 2\cos\theta\sin\theta)$$

For max/min R, $\frac{dR}{d\theta} = 0$.

$$5\sqrt{2}\left(\cos 2\theta + \sin 2\theta\right) = 0$$

$$\cos 2\theta + \sin 2\theta = 0$$

$$\cos 2\theta + \sin 2\theta = R\cos(2\theta - \alpha)$$

$$= R(\cos 2\theta\cos\alpha + \sin 2\theta\sin\alpha)$$

$1 = R\cos\alpha$ — (1)

$1 = R\sin\alpha$ — (2)

$\frac{(2)}{(1)}$: $\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$. $\quad R = \sqrt{a^2+b^2}$

$= \sqrt{4} = \underline{\sqrt{2}}$

$\therefore \cos 2\theta + \sin 2\theta = \sqrt{2}\cos\left(2\theta - \frac{\pi}{4}\right) = 0$

$\sqrt{2}\cos\left(2\theta - \frac{\pi}{4}\right) = 0$; $\cos\left(2\theta - \frac{\pi}{4}\right) = 0$. ✓

$2\theta - \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}$ $\left(\frac{\pi}{4} < \theta < \frac{\pi}{2}\right)$

$\theta = \frac{3\pi}{8}, \cancel{\frac{7\pi}{4}}$ $\left(\frac{\pi}{4} < \theta < \frac{\pi}{2}\right)$

Determine nature ✓

$$\frac{d^2R}{d\theta^2} = 5\sqrt{2}\left(-2\sin 2\theta + 2\cos 2\theta\right)$$

12

At $\theta = \frac{3\pi}{8}$, $\frac{d^2R}{d\theta^2} < 0$

$\therefore$ R is max when $\theta = \frac{3\pi}{8}$. ✓

v.) when $\theta = \frac{3\pi}{8}$, $R = 5\sqrt{2}\left(\cos\frac{3\pi}{8}\sin\frac{3\pi}{8} - \cos^2\frac{3\pi}{8}\right)$

$= 1.46\text{m}$. But mouse is sitting 1.8m up slope

$\therefore$ the cat will need to run up the slope 0.34m to catch the mouse. ~~0.34m~~ ✓