International Grammar School

# Year 12
# Mathematics Extension 1
# HSC Assessment Task 3
# 2011

**General Instructions**

- Reading time – 5 minutes
- Working time – 1 hour
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question

**Note:** Any time you have remaining should be spent revising your answers.

**Total marks – 36**

- Attempt Questions 1 – 3
- All questions are of equal value
- Start each question in a new writing booklet
- Write your name on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a booklet marked with your name and "N/A" on the front cover

**DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM**

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

**Question 2 (12 marks)** Use a SEPARATE writing booklet. **Marks**

a) Let $f(x) = x^3 + 2x^2 + 5x - 4$. The equation $f(x)$ has only one real root.

i) Explain why the root lies between $x = 0$ and $x = 1$. 2

ii) Use one application of the "halving the interval" method to find a better estimation of the root. 2

iii) Use your answers above to describe the interval in which the root will be found, justifying your answer. 1

b) i) Explain why the equation $\log_e x + x^2 - 4x + 1 = 0$ has a root between $x = 3$ and $x = 4$. You must refer to the domain of $y = \log_e x + x^2 - 4x + 1$ in your answer. 2

ii) Let $x = 3.4$ be a first approximation to the root. Apply Newton's method to obtain a better approximation to the root correct to two decimal places. 3

c) Prove: 2

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d^2x}{dt^2}$$

**End of Question 2**





**Question 3 (12 marks)** Use a SEPARATE writing booklet. **Marks**

a) The acceleration, $\ddot{x}$, of a particle in terms of its displacement, $x$ metres from a point O is given by $\ddot{x} = -e^{-2x}$. Initially the particle is at the origin and the velocity $v$ of the particle is $v = 1\ m/s$. Find:

i) an expression for the velocity of the particle in term of $x$. 3

ii) displacement as a function of time. 2

b) A spherical bubble is expanding so that its volume increases at a constant rate of $50 mm^3$ per second. At what rate is the surface area of the bubble increasing when the radius is 2 cm? 3

c) A particle P is moving in a straight line with its position in metres from a fixed origin at a time $t$ seconds being given by

$$x = 1 + 4\cos\left(2t - \frac{\pi}{6}\right).$$

i) Show that P is moving in simple harmonic motion. 2

ii) What is the amplitude and centre of the motion? 1

iii) What is the maximum speed of the particle? 1

**END OF ASSESSMENT TASK**

HSC 2011 3U AT3.

2

Q1.

a) i) $\int \cos^2 9x \, dx$.

$\cos 18x = \cos^2 9x - \sin^2 9x$
$= 2\cos^2 9x - 1$
$\cos^2 9x = \frac{1}{2}(\cos 18x + 1)$

$= \frac{1}{2}\int \cos 18x + 1 \, dx$ ✓

$= \frac{1}{2}\left[\frac{\sin 18x}{18} + x\right] + C$ ✓

ii) $y = \sqrt{2}\sin x \quad 0 \le x \le \frac{\pi}{12}$.

$V = \pi \int_0^{\frac{\pi}{12}} y^2 \, dx$

$= \pi \int_0^{\frac{\pi}{12}} 2\sin^2 x \, dx$ ✓ $\quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$= 2\frac{\pi}{2} \int_0^{\frac{\pi}{12}} 1 - \cos 2x \, dx$ ✓ If in

$= \pi \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/12}$

$= \pi \left[\frac{\pi}{12} - \frac{\sin \pi/6}{2} - 0\right]$

$= \pi \left[\frac{\pi}{12} - \frac{1}{4}\right] u^3$ ✓

b) i) $\sin x - \cos 2x = 2\sin^2 x + \sin x - 1$

LHS: $\sin x - \cos 2x = \sin x - (1 - 2\sin^2 x)$ ✓
$= 2\sin^2 x + \sin x - 1$

ii) $\sin x - \cos 2x = 0 \quad 0 \le x \le 2\pi$

ie $2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$ ✓

$\sin x = \frac{1}{2}$ Q1, Q2 $\quad \sin x = -1$

$x = \frac{3\pi}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$. ✓✓

c) $\int \frac{\sin 2x}{1 + \sin^2 x} \, dx$

$u = \sin^2 x$
$du = 2\cos x \sin x \, dx$
$= \sin 2x \, dx$

$= \int \frac{du}{1+u}$ ✓

$= \ln(1+u) + C$ ✓

$= \ln(1 + \sin^2 x) + C$. ✓

Question 2.

a) $f(x) = x^3 + 2x^2 + 5x - 4$.

i) $f(0) = -4 < 0$

$f(1) = 4 > 0$

since sign changes and $f(x)$ continuous $0 \leq x \leq 1$, root exists

~~iii)~~ $x_1 = \frac{1}{2}$.

$f(x_1) = -\frac{7}{8}$

$= -0.875.$

~~iii)~~ since $f(\frac{1}{2})$ is closer to zero, $x_1$ ~~is~~ a better estimate

iii) $f(\frac{1}{2}) < 0$, so interval is $\frac{1}{2} < x < 1$.

b) i) $f(x) = \log_e x + x^2 - 4x + 1 \quad 3 \leq x \leq 4$

$f(3) = -0.90138.. < 0$

$f(4) = 2.38629.. > 0$

since $f(x)$ is continuous for $3 \leq x \leq 4$ (since the domain for $f(x)$ is $> 0$) and the sign changes, a root exists in the interval.

ii) $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f'(x) = \frac{1}{x} + 2x - 4$

so, $x_2 = 3.4 - \frac{f(3.4)}{f'(3.4)}$

$= 3.34060...$

$= 3.34 \quad 2\text{ dp}$



$x_3 = 3.34 - \frac{f(3.34)}{f'(3.34)}$

$= 3.339472...$

$= 3.34 \quad 2\text{ dp}.$

must do this step for the degree of accuracy of 2 dp.

$\therefore f(3.34) = 0.00157....$

which is closer to zero

$\therefore x = 3.34$ is a better approximation.

c) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dv}\left(\frac{1}{2}v^2\right) \cdot \frac{dv}{dx}$

$= v \cdot \frac{dv}{dx}$

$= \frac{d\cancel{x}}{dt} \cdot \frac{dv}{d\cancel{x}}$

$= \frac{dv}{dt}$

$= \frac{d^2x}{dt^2}$

SK 2011 HSC AT3 5 6.

Q3

a) $\ddot{x} = -e^{-2x}$. $t=0$ $x=0$ $v = 1$ m/s.

i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -e^{-2x}$ ✓

$\frac{1}{2}v^2 = \int -e^{-2x}\,dx$

$= \frac{1}{2}\int -2e^{-2x}\,dx$

$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x} + C.$

$x=0$ $v=1$.

$\frac{1}{2} = \frac{1}{2}e^0$ $\therefore C = 0$

$\frac{1}{2}v^2 = \frac{1}{2}e^{-2x}$ ✓

$v^2 = e^{-2x}$

$v = \pm\sqrt{e^{-2x}}$.

$v = 1$ when $x = 0$ which is only satisfied by $v = \sqrt{e^{-2x}}$ ie $v > 0$.

so $v = \sqrt{e^{-2x}} = e^{-x}$ ✓

ii) $v = \frac{dx}{dt} = \left(e^{-2x}\right)^{1/2}$

$= e^{-x} = \frac{1}{e^x}$

$\frac{dt}{dx} = e^x$ ✓

$t = \int e^x\,dx$

$= e^x + C.$

$t=0$ $x=0$.

$0 = e^0 + C$ $C = -1$.

so $t = e^x - 1$

$t + 1 = e^x$

$x = \ln(t+1)$ ✓

b) $\frac{dV}{dt} = 50$ mm$^3$/s.

want $\frac{dA}{dt}$ when $r = 2$ cm or $20$ mm

$\frac{dA}{dt} = \frac{dV}{dt} \times \frac{dA}{dV}$

$= \frac{dV}{dt} \times \frac{dA}{dr} \times \frac{dr}{dV}$ ✓

$V = \frac{4}{3}\pi r^3$ $\frac{dV}{dr} = 4\pi r^2$

$A = 4\pi r^2$ $\frac{dA}{dr} = 8\pi r$ ✓

$\frac{dA}{dt} = 50 \times \cancel{8}^{2}\cancel{\pi}\cancel{r} \times \frac{1}{\cancel{4}\cancel{\pi} r^{\cancel{2}}}$

$= \frac{100}{r}$

$= \frac{100}{20}$ $r = 20$ mm

$= 5$ mm$^2$/s. ✓