Polynomials

Name: ______________________ Total: /10

Of the three roots of the cubic equation $x^3 - 15x + 4 = 0$, two are reciprocals.

1. Find the other root.

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2. Find all the roots and verify that two of them are recipricols.

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The cubic polynomial equation $x^3 = ax^2 + bx + c$ has three real roots, two of which are opposites. Prove that

3. One of the roots is a.

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4. The other roots are $\sqrt{b}$ and $-\sqrt{b}$.

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5. $ab + c = 0$.

$f(x) = x^4 + 4x^3 + 8x - 4$

6. Show that $f(x)$ has a zero, α, between 0 and 1.

7. Determine whether α lies closer to 0 or to 1.

8. Taking 0.5 as a first approximation, use Newton's method to find a two-placed decimal approximation to α.

9. Show by division, or otherwise, that $(x^2 + 2)$ is a factor of $f(x)$.

10. Show that $f(x)$ has only two real roots and find the value of α correct to three decimal places.

Polynomials

1. $x^3 - 15x + 4 = 0$

Let the roots be $\alpha, \frac{1}{\alpha}, \beta$

$$\alpha + \frac{1}{\alpha} + \beta = 0$$

$$1 + \alpha\beta + \frac{\beta}{\alpha} = -15$$

$$\beta = -4 \checkmark$$

~~$\therefore \alpha + \frac{1}{\alpha} = 4$~~

~~$\alpha^2 + 1 - 4\alpha = 0$~~

~~$\alpha^2 - 4\alpha + 1 = 0$~~

~~$\alpha = \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 1}}{2}$~~

~~$= \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$~~

2. ~~[illegible]~~ $1 - 4\alpha - \frac{4}{\alpha} = -15$

$$\alpha - 4\alpha^2 - 4 = -15\alpha$$

$$4\alpha^2 - 16\alpha + 4 = 0 \checkmark$$

$$\alpha^2 - 4\alpha + 1 = 0$$

$$\alpha = \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 1}}{2} = \frac{4 \pm \sqrt{12}}{2}$$

$$= 2 \pm \sqrt{3} \checkmark$$

One of the roots is $2 + \sqrt{3}$

other root: $\frac{1}{2 + \sqrt{3}} \cdot \frac{2 - \sqrt{3}}{2 - \sqrt{3}}$

$$= \frac{2 - \sqrt{3}}{4 - 3} = 2 - \sqrt{3} \checkmark$$

3. $x^3 - ax^2 - bx - c = 0$

Let the roots be $\alpha, -\alpha, \beta$

$$\alpha - \alpha + \beta = a$$

$$\beta = a \checkmark$$

4. $-\alpha^2 + \alpha\beta - \alpha\beta = -b \checkmark$

$$-\alpha^2 = -b$$

$$\alpha^2 = b$$

$$\alpha = \pm\sqrt{b} \checkmark$$

5. $-\alpha^2\beta = c$

$$-b \cdot a = c \checkmark$$

$$-ab = c$$

$$ab + c = 0 \checkmark$$

6. $f(0) = -4$

$f(1) = 9$

$f(0) < 0$ ✓

$f(1) > 0$

∴ α lies between 1 & 0 ✓

7. [illegible] α is closer to 0 ✓

8. $f(x) = x^4 + 4x^3 + 8x - 4$

$f(0.5) = 0.5625$

$f'(x) = 4x^3 + 12x^2 + 8$ ✓

$= 11.5$

$z_2 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$= 0.45$ ✓

9. ~~$x^2+2 \,\overline{)\, x^4 + 4x^3 + 8x - 4}$ quotient $x^2 + 4x + 2$~~

~~$-x^4 + 2 \quad + 2x^2 \quad + 8x$~~

~~$-4x^3 - 2x^2 - 4$~~

10. $\Delta = b^2 - 4ac$

$= 8$

∴ two real roots

~~10b. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$~~

~~$= \frac{-4 \pm \sqrt{8}}{2}$~~

~~$= -.586$ ✓~~

~~$= -3.41$ ✓~~

(10)(b) For $f(x) = (x^2+2)(x^2+4x-2) = 0$

$x = \frac{-4 \pm \sqrt{16+8}}{2}$

$= 0.449$ or -4.449

∴ $\underline{\alpha = 0.449 \text{ (to 3dp)}}$.

$$\begin{array}{r|l} & x^2 + 4x - 2 \\ \hline x^2+0x+2 & x^4 + 4x^3 + 0x^2 + 8x - 4 \\ & x^4 + 0x^3 + 2x^2 \\ \hline & 4x^3 - 2x^2 + 8x \\ & 4x^3 + 0x^2 + 8x \\ \hline & -2x^2 + 0x - 4 \\ & -2x^2 + 0x - 4 \\ \hline & 0 \end{array}$$

Polynomials

Amy . Ow

EXCELLENT

① let $P(x) = x^3 - 15x + 4$

let roots be $\alpha, \frac{1}{\alpha}, \beta$ ✓

$\alpha + \frac{1}{\alpha} + \beta = \frac{-b}{a} = 0$ — ①

[illegible]

$\alpha \times \frac{1}{\alpha} \times \beta = \frac{-d}{a}$ ✓

$\beta = -4$ ∴ other root is -4.

② Sub $\beta = -4$ in ①

$\Rightarrow \alpha + \frac{1}{\alpha} - 4 = 0$

~~$4\alpha^2 - 16\alpha + 4 = 0$~~

$\alpha^2 - 4\alpha + 1 = 0$ ✓

$\therefore \alpha = \frac{4 \pm \sqrt{16-4}}{2}$

$= \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

∴ roots are $(2+\sqrt{3}), (2-\sqrt{3}), -4$ ✓

$\frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$

∴ two roots are reciprocals. ✓

③ $x^3 - ax^2 - bx - c = 0$

let roots be $\alpha, -\alpha, \beta$

$\alpha + (-\alpha) + \beta = a$

$\therefore \beta = a$ ✓

∴ one of the roots is a ✓

④ $-\alpha^2 + \alpha\beta - \alpha\beta = -b$ ✓

$\therefore \alpha^2 = b$

$\therefore \alpha = \sqrt{b}, -\sqrt{b}$ ✓

⑤ $-\alpha^2\beta = c$

$\therefore c = -\alpha^2\beta$ ✓

$ab + c = 0$

$LHS = \beta(\alpha^2) + (-\alpha^2\beta)$

$= 0$ ✓

$= \underline{RHS}$.

~~⑥ $f(x) = x^4 + 4x^3 + 8x - 4$~~

~~$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\theta + \beta\delta + \gamma\delta = 0$~~