Revision Exercise

1. Find the values of A, B and C such that
$$2x^3 + x^2 - 5x + 7 \equiv (x + 2)(Ax^2 + Bx + C) + 5$$

2. Find $F(x)$ if $6x^4 + 11x^3 + 8x + 5 \equiv (2x + 1)F(x)$.

3. Express $\dfrac{9x}{(2x + 1)^2(1 - x)}$ in partial fractions.

4. Express $\dfrac{x - 2}{(x^2 + 1)(x - 1)^2}$ in partial fractions in its simplest form.

5. Express the function $\dfrac{7x + 4}{(x - 3)(x + 2)^2}$ as the sum of partial fractions with constant numerators.

6. If $f(x) = x^6 - 5x^4 - 10x^2 + p$, find the value of p such that $(x - 1)$ is a factor of $f(x)$. With this value of p, find another factor of $f(x)$ in the form $(x + a)$, where a is a constant.

7. If $f(x) = ax^2 + bx + c$ has remainder 1, 25 and 1 when divided by $(x - 1)$, $(x + 1)$ and $(x - 2)$ respectively, show that the function $f(x)$ is a perfect square.

8. Given that $(x - 3)$ and $(2x + 1)$ are factors of
$$f(x) \equiv ax^4 + bx^3 + 13x^2 + 30x + 9,$$
find the values of a and b.
With these values of a and b, show that $f(x) \geqslant 0$ for all $x \in \mathbb{R}$.

9. The polynomial $P(x)$ has remainder 1 when divided by $(x - 1)$, and remainder 3 when divided by $(x + 1)$. Find the remainder when $P(x)$ is divided by $(x^2 - 1)$.

10. Find the value of k such that $(2x - 1)$ is a factor of
$$f(x) \equiv 4x^4 + 8x^3 + kx^2 - 11x + 6$$
Assuming this value of k, factorise $f(x)$ completely.

11. When the polynomial $P(x)$ is divided by $(x^2 - 1)$, its remainder is $(ax + b)$, where a and b are constants. Given that $(x + 1)$ is a factor of $P(x)$, and that the remainder is 4 when $P(x)$ is divided by $(x - 1)$, find the values of a and b.

12. If the function $ax^2 + bx + c$ has a minimum value -5 when $x = -1$ and 0 when $x = -2$, find the values of a, b and c. Find the range of values of x such that $ax^2 + bx + c > 75$.

13. Show that the expression $-x^2 + px + q < 0$ for all values of $x \in \mathbb{R}$ if $-q > \frac{1}{4}p^2$. Prove, also, that $(24 - x)(x - 8) - k < 0$ for all values of $x \in \mathbb{R}$ if $k > 64$. Hence, or otherwise show that the expression
$$(6 + y)(4 - y)(y + 4)(y - 2) - 65 < 0$$
for all values of $y \in \mathbb{R}$.

14. If $x = 2$ is a root of the equation
$$a^2x^2 + 2(2a - 5)x + 8 = 0,$$
find the possible values of a.
Find the corresponding roots with these values of a.

15. (a) Show that the equation $x^2 + (3\alpha - 2)x + \alpha(\alpha - 1) = 0$ has real roots for all values of $\alpha \in \mathbb{R}$.
(b) Show that $x^2 - x + 1$ has the same sign for all values of x.

16. Show that roots of the equation
$$px^2 + (p + q)x + q = 0$$
are real for all values of p and q.

17. Find the value of p if $x^2 + (p + 3)x + 2p + 3$ is an expression in the form of a perfect square.

18. If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, express $(\alpha - 2\beta)(2\alpha - \beta)$ in terms of a, b and c. Deduce the condition that one root of the equation is twice the other root.

19. If a, b and $c \in \mathbb{R}$, with $a \neq 0$, and the roots of the quadratic equation $ax^2 + bx + c = 0$ are real, show that the roots of $a^2y^2 - (b^2 - 2ac)y + c^2 = 0$ are also real.
If the roots of the quadratic equation $ax^2 + bx + c = 0$ are α and β, state the values of $\alpha + \beta$ and $\alpha\beta$ in terms of a, b and c. Hence, find the roots of the second equation in terms of α and β.

20. The roots of the equation $x^2 + px + 1 = 0$ are α and β. If one of the roots of the equation $x^2 + qx + 1 = 0$ is α^3, prove that the other root is β^3.
Without solving any equation, show that $q = p(p^2 - 3)$. Obtain the quadratic equation with roots α^9 and β^9, giving the coefficients of x in terms of q.

21. If α and β are the roots of the equation $x^2 + px + q = 0$, show that p and q are the roots of the equation $x^2 + (\alpha + \beta - \alpha\beta)x - \alpha\beta(\alpha + \beta) = 0$. Find the non-zero values of p and q if the roots of the second equation are
(a) α and β,
(b) α^2 and β^2.

22. The equation $x^2 + px + q = 0$ has roots whose difference is $2\sqrt{3}$ and product 6. Find the possible values of p and q.

23. The equation $px^2 + qx + r = 0$, where $p \neq 0$, has roots α and β. Obtain a quadratic equation with roots $\alpha^2 + \frac{1}{\beta^2}$ and $\beta^2 + \frac{1}{\alpha^2}$, giving its coefficients in terms of p, q and r.

24. The polynomial $2x^4 - ax^3 + 19x^2 - 20x + 12$ has a factor in the form $(x - k)^2$, where $k \in \mathbb{N}$. Find the values of k and a and show that the polynomial is non-negative for all $x \in \mathbb{R}$.

25. (a) Find the value of the constant k such that $(x + 1)$ is a factor of $2x^3 + 7x^2 + kx - 3$. With this value of k, solve the equation
$$2x^3 + 7x^2 + kx - 3 = 0$$
(b) Find the numerical values of the constants A, B, C and D such that
$$x^4 + Ax^3 + 5x^2 + 3 \equiv (x^2 + 4)(x^2 - x + B) + Cx + D.$$

26. The roots of the quadratic equation $x^2 + px + q = 0$, where $q \neq 0$, are α, β and one root of the quadratic equation $x^2 + p'x + q = 0$ is $k\alpha$. Show that the other root of the equation is $\frac{\beta}{k}$.
By assuming that $k^2 \neq 1$, write down an expression for the sum of the roots for each of the quadratic equations. Hence, find α and β in terms of p, p' and k. Deduce that $k(kp - p')(kp' - p) = (k^2 - 1)^2q$.
For any p, p' and q, show that the sum of the four possible values of k is $\frac{pp'}{q}$.

27. If $f(x) = x^4 + 2x^3 + 5x^2 - 16x - 20$, show that $f(x)$ can be expressed in the form $(x^2 + x + a)^2 - 4(x + b)^2$, where a and b are constants to be determined.
Hence, or otherwise, find both the real roots of the equation $f(x) = 0$. Find also the set of values of x such that $f(x) > 0$.

28. By using the substitution $y = x + \frac{1}{x}$, change the equation $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ into a quadratic equation in y. Determine the two values of y which satisfy this quadratic equation. Hence, solve the equation $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$.

29. (a) Given that the roots of $ax^2 + bx + c = 0$ are β and $n\beta$, show that $(n + 1)^2ac = nb^2$.
(b) Given that α and β are the roots of the equation $x^2 - px + q = 0$, prove that $\alpha + \beta = p$ and $\alpha\beta = q$. Prove also, that
(i) $\alpha^{2n} + \beta^{2n} = (\alpha^n + \beta^n)^2 - 2q^n$,
(ii) $\alpha^4 + \beta^4 = p^4 - 4p^2q + 2q^2$.
Hence, form a quadratic equation whose roots are the fourth power of the roots of $x^2 - 3x + 1 = 0$.

30. Determine the condition to be satisfied by k such that the expression $2x^2 + 6x + 1 + k(x^2 + 2)$ is positive for all $x \in \mathbb{R}$.

31. Find the set of values of x which satisfy the following inequalities.

(a) $\dfrac{x(x+2)}{x-3} < x + 1$ (b) $\dfrac{x^2 + 4x - 3}{x^2 + 1} < x$

32. Without using tables or calculator, show that

(a) $3 < \sqrt{13} < 4$, and deduce that $0 < \sqrt{13} - 3 < 1$,

(b) $(\sqrt{13} + 3)^4 + (\sqrt{13} - 3)^4 = 1904$,

(c) $1903 < (\sqrt{13} + 3)^4 < 1904$

33. Express $(\sqrt{p} + q\sqrt{r})^2$ in the form $a + b\sqrt{c}$. Without evaluating the square root or using tables or calculator, show that $\sqrt{10} + 2\sqrt{2} < 6$.

34. If p and q are positive numbers, prove that

$$(1-p)(1-q) > 1 - p - q.$$

If $p, q, r \in \mathbb{R}^+$, with at least one of them less than unity, prove that

$$(1-p)(1-q)(1-r) > 1 - p - q - r.$$

35. Solve each of the following inequalities.

(a) $2|x + 2| < |4 - x|$

(b) $|3x + 1| - 4|x + 1| \geqslant 0$

(c) $|x^2 - 3x - 2| < 2$

(d) $\dfrac{6}{|x| + 1} < |x|$

36. Solve the following simultaneous equations.

$$\frac{x}{3} + \frac{y}{4} = 1 \text{ and } \frac{3}{x} - \frac{2}{y} = \frac{7}{12}$$

37. Find, in terms of a, b and c, the values of x, y and z, if

$$3x - y - z = a$$
$$-x + 3y - z = b$$
$$-x - y + 3z = c$$

38. From the equations

$$11x^2 - 8xy + 5y^2 = 32$$

and

$$x^2 + y^2 = 8,$$

deduce that $7x^2 - 8xy + y^2 = 0.$

Hence find all pairs of values of x and y which satisfy the given equations.

39. By using the substitution $y = x + \dfrac{1}{x}$, solve the equation $2x^4 + x^3 - 6x^2 + x + 2 = 0$.

40. Find all pairs of values of x and y such that $\dfrac{3x + y + 1}{8} = \dfrac{x - y}{5} = \dfrac{x^2 - y^2}{5}$.

Revision Exercise

1. $A = 2$, $B = -3$, $C = 1$
2. $3x^3 + 4x^2 - 2x + 5$
3. $\frac{1}{1-x} + \frac{2}{2x+1} - \frac{3}{(2x+1)^2}$
4. $\frac{1}{x-1} - \frac{1}{2(x-1)^2} - \frac{2x+1}{2(x^2+1)}$
5. $\frac{1}{x-3} - \frac{1}{x+2} + \frac{2}{(x+2)^2}$
6. 14, $x + 1$
8. $a = 4$, $b = -20$
9. $2 - x$
10. $k = -7$, $(2x-1)(x-1)(x+2)(2x+3)$
11. $a = 2$, $b = 2$
12. $a = 5$, $b = 10$, $c = 0$; $x < -5$ or $x > 3$
14. $a = 1$, $x = 2, 4$; $a = -3$, $x = 2, \frac{4}{9}$
17. $3, -1$
18. $\frac{(2b^2 - 9ac)}{a^2}$, $2b^2 = 9ac$
19. $-\frac{b}{a}, \frac{c}{a}, \alpha^2, \beta^2$
20. $x^2 + q(q^2 - 3)x + 1 = 0$
21. (a) $p = 1$, $q = -2$, (b) $p = q = 4$
22. $p = \pm 6$, $q = 6$
23. $p^2r^2x^2 - (q^2 - 2pr)(p^2 + r^2)x + (p^2 + r^2)^2 = 0$
24. $k = 2$, $a = 10$
25. (a) $2; \frac{1}{2}, -1, -3$ (b) $-1, 1, 5, -1$
26. $\alpha = \frac{kp' - p}{1 - k^2}$, $\beta = \frac{k(p' - kp)}{k^2 - 1}$
27. $a = 4$, $b = 3$; $x = -1, 2$; $\{x : x < -1 \text{ or } x > 2\}$
28. $y = \frac{10}{3}, -\frac{5}{2}$; $x = -\frac{1}{2}, -2, \frac{1}{3}, 3$
29. (b) $x^2 - 47x + 1 = 0$
30. $k > 1$
31. (a) $-\frac{3}{4} < x < 3$ (b) $-\sqrt{3} < x < 1$ or $x > \sqrt{3}$.
33. $p + q^2r + 2q\sqrt{pr}$
35. (a) $-8 < x < 0$ (b) $-3 \leqslant x \leqslant -\frac{5}{7}$
    (c) $-1 < x < 0$, $3 < x < 4$
    (d) $x < -2$, $x > 2$
36. $x = 9$, $y = -8$; $x = \frac{12}{7}$, $y = \frac{12}{7}$
37. $x = \frac{1}{4}(2a + b + c)$, $y = \frac{1}{4}(a + 2b + c)$,
    $z = \frac{1}{4}(a + b + 2c)$
38. $x = \pm 2$, $y = \pm 2$; $x = \pm\frac{2}{5}$, $y = \pm\frac{14}{5}$
39. $x = 1, 1, -2, -\frac{1}{2}$
40. $x = -\frac{1}{4}$, $y = -\frac{1}{4}$; $x = 3$, $y = -2$