ST CATHERINES – MAR 2016 – EXT2

Polynomials

Due: Week 4 Wednesday

Question 1

(a) (i) Use De Moivre's Theorem to write $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. **3**

(ii) Hence show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$. **2**

(iii) Hence, show that the roots of the equation $x^4+4x^3-6x^2-4x+1=0$ are $x=\tan\dfrac{\pi}{16}$, $x=\tan\dfrac{5\pi}{16}$, $x=\tan\dfrac{9\pi}{16}$ and $x=\tan\dfrac{13\pi}{16}$. **3**

(iv) Hence prove that $\tan^2\dfrac{\pi}{16}+\tan^2\dfrac{3\pi}{16}+\tan^2\dfrac{5\pi}{16}+\tan^2\dfrac{7\pi}{16}=28$. **3**

Question 2

(a) Consider the polynomial equation $x^4+ax^3+bx^2+cx+d=0$, where a, b, c, and d are all integers. The equation has a non-real root of the form qi where q is a real number and $q\neq 0$.

(i) Explain why $-qi$ is also a root of the equation. **1**

(ii) Show that $c=q^2a$ by considering the relationships between the roots and coefficients of the polynomial. **2**

(iii) Show that $c^2+a^2d=abc$. **2**

(iv) If 2 is also a root of the equation, and $b=0$, show that c is even. **2**

Question 3

A polynomial $P(x)=4x^3-3x-1$ has roots α, β and δ.

Which of the following polynomials has roots $\dfrac{1}{\alpha+\beta}$, $\dfrac{1}{\alpha+\delta}$ and $\dfrac{1}{\beta+\delta}$?

(A) $Q(x)=x^3-3x^2-4$

(B) $Q(x)=x^3-3x^2+4$

(C) $Q(x)=x^3+3x^2-4$

(D) $Q(x)=x^3+3x^2+4$

Question 4

The polynomial $P(x)$ is defined as $P(x)=(x^2+4)^2(2x+1)(x-a)$ where a is a real number.

Which of the following statements is FALSE?

(A) $P(x)$ must have real coefficients.

(B) $P(x)$ must have 6 roots over the complex field.

(C) $P(x)$ must have integer coefficients.

(D) $P(x)$ must be non-monic.

Question 5

The equation $x^4-2x^2-5x+3=0$ has roots α,β,γ and δ.

(i) Find the value of $\alpha^2+\beta^2+\gamma^2+\delta^2$. **2**

(ii) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. **1**

(iii) State the number of real and non-real roots of the equation, giving reasons for your answer. **2**

Assignment Polynomials

1\. a) i. $\cos 4\theta + i\sin 4\theta = c^4 + 4c^3 is + 6c^2(is)^2 + 4c(is)^3 + (is)^4$

$= (c^4 - 6c^2s^2 + s^4) + i(4c^3s - 4cs^3)$

$\boxed{c = \cos\theta \quad s = \sin\theta}$

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

ii. $\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta}$

$= \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

$= \dfrac{4\frac{\sin\theta}{\cos\theta} - 4\frac{\sin^3\theta}{\cos^3\theta}}{1 - 6\frac{\sin^2\theta}{\cos^2\theta} + \frac{\sin^4\theta}{\cos^4\theta}}$

$\therefore \tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

iii. $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$1 - 6x^2 + x^4 = 4x - 4x^3$

$1 = \dfrac{4x - 4x^3}{1 - 6x^2 + x^4}$

If $x = \tan\theta$, $\tan 4\theta = 1$

$\tan 4\theta = 1$

$4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$

$\therefore$ the roots are:

$x = \tan\frac{\pi}{16}$, $x = \tan\frac{5\pi}{16}$, $x = \tan\frac{9\pi}{16}$, $x = \tan\frac{13\pi}{16}$

iv. $\tan\frac{9\pi}{16} = \tan\left(-\frac{7\pi}{16}\right)$

$\tan\frac{13\pi}{16} = \tan\left(-\frac{3\pi}{16}\right)$

let the roots of $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ be $x = \alpha, \beta, \gamma, \sigma$

$\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = \alpha^2 + \beta^2 + \gamma^2 + \sigma^2$

$= (\alpha+\beta+\gamma+\sigma)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\sigma + \beta\gamma + \beta\sigma + \gamma\sigma)$

$= (-4)^2 - 2(-6)$

$= 16 + 12$

$= 28$

2\. a) i. If $(x - qi)$ is a factor of $x^4 + ax^3 + bx^2 + cx + d = 0$ where a, b, c and d are real coefficients, $(x + qi)$ is also a factor as it is its complex conjugate.

$\therefore -qi$ is a root of the eqn.

ii. let the roots be $x = \alpha, \beta, qi, -qi$

$c = \alpha\beta qi - \alpha\beta qi + \alpha q^2 + \beta q^2$

$= q^2(\alpha + \beta)$

$a = \alpha + \beta + qi - qi$

$= \alpha + \beta$

$\therefore c = q^2 a$

iii. $a = \alpha+\beta$

$b = \alpha\beta + \alpha q i - \alpha q i + \beta q i - \beta q i - q^2 i^2$

$= \alpha\beta + q^2$

$c = q^2(\alpha+\beta)$

$d = -\alpha\beta q^2 i^2$

$= \alpha\beta q^2$

LHS $= c^2 + a^2 d$

$= q^4(\alpha+\beta)^2 + (\alpha+\beta)^2\alpha\beta q^2$

$= q^2(\alpha+\beta)^2(q^2+\alpha\beta)$

RHS $= abc$

$= (\alpha+\beta)(\alpha\beta+q^2)q^2(\alpha+\beta)$

$= q^2(\alpha+\beta)^2(q^2+\alpha\beta)$

$=$ LHS

$\therefore c^2 + a^2 d = abc$

iv. $c^2 + a^2 d = 0$

$c^2 + a^2(2\beta q^2) = 0$

$c^2 + a^2(2\beta \cdot \frac{c}{a}) = 0$

$c^2 + 2a\beta c = 0$

$c(c + 2a\beta) = 0$

$c = -2a\beta$

$\therefore$ c is even

3. $\alpha+\beta+\gamma = -\frac{b}{a}$

$= 0$

$\therefore \alpha+\beta = -\gamma$

$\beta+\gamma = -\alpha$

$\alpha+\gamma = -\beta$

$\therefore$ the roots are : $\frac{1}{-\gamma}, \frac{1}{-\beta}, \frac{1}{-\alpha}$

consider $P(-\frac{1}{x}) : P\left(\frac{-1}{-\frac{1}{\alpha}}\right)$

$= P(\alpha) = 0$

$\therefore P(-\frac{1}{x}) = 4(-\frac{1}{x})^3 - 3(-\frac{1}{x}) - 1 = 10,$

$0 = \frac{-4}{x^3} + \frac{3}{x} - 1$

$0 = -x^3 + 3x^2 - 4$

$\therefore Q(x) = x^3 - 3x^2 + 4$ $\boxed{B}$

4. $\boxed{C}$

5. i. $\alpha^2+\beta^2+\gamma^2+\sigma^2 = (\alpha+\beta+\gamma+\sigma)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\sigma+\beta\gamma+\beta\sigma+\gamma\sigma)$

$= 0^2 - 2(-2)$

$= 4$

ii. $x^4 - 2x^2 - 5x + 3 = 0$

$\alpha^4 - 2\alpha^2 - 5\alpha + 3 = 0$

$\beta^4 - 2\beta^2 - 5\beta + 3 = 0$ +

$\gamma^4 - 2\gamma^2 - 5\gamma + 3 = 0$

$\sigma^4 - 2\sigma^2 - 5\sigma + 3 = 0$

$\alpha^4+\beta^4+\gamma^4+\sigma^4 - 2(\alpha^2+\beta^2+\gamma^2+\sigma^2) - 5(\alpha+\beta+\gamma+\sigma) + 12 = 0$

$\alpha^4+\beta^4+\gamma^4+\sigma^4 = 2(\alpha^2+\beta^2+\gamma^2+\sigma^2) + 5(\alpha+\beta+\gamma+\sigma) - 12$

$= 2(4) + 5(0) - 12$

$= -4$

iii. since $\alpha^2+\beta^2+\gamma^2+\sigma^2$ is negative,
there are at least 2 complex roots (coefficients are real)

since 3 is the constant, ~~the~~ for there to be real roots, 1, -1, 3 or -3 must 0.

let the eqn $= P(x)$

$P(1) = -3$

$P(-1) = 7$

$P(3) = 51$

$P(-3) = 81$

$\therefore$ there are no real roots
↳ only 4 complex roots