PORT HACKING HIGH SCHOOL

MATHEMATICS

2 UNIT

YEAR 12 HALF YEARLY EXAMINATION

2009

Value: 25% of HSC Assessment

Time: 2 hours

Instructions:

- Attempt all questions.
- Show all working.
- Start each of the 7 questions on a new page.
- Write on one side of the paper only.
- Write your name on each page.
- Staple together the pages where more than one page is used for a question.

Handing in the paper.

- Hand in your question paper.
- Hand in each of the 7 questions separately.

QUESTION 1 *12 marks*

Ba

Marks

(a) Simplify $\frac{x}{x^2-4}+\frac{2}{x-2}$ 3

(b) Solve $|x-5|=3$ 2

(c) Evaluate, correct to 3 significant figures $\sqrt{\frac{3^2+12^2}{231-12^2}}$ 1

(d) Solve $x^2=5x$ 2

(e) Find integers a and b such that $(3-\sqrt{2})^2=a-b\sqrt{2}$ 2

(f) Evaluate the following limits. 2

i) $\lim_{x\to 5}\frac{x^3-125}{x-5}$

ii) $\lim_{x\to\infty}\frac{x^2+6x-4}{x^3-5}$

QUESTION 2 Start a new page *12 marks*

Marks

(a) The points A(1, 6), B(6, -2) and C(-1, 1) are vertices of a triangle.

i) Find the length of the interval BC. *2*

ii) Find the equation of the line BC. *2*

iii) Find the exact perpendicular distance from A to BC. *2*

iv) Hence or otherwise, find the area of the triangle ABC. *1*

v) Write down the coordinates of D, such that ADBC is a parallelogram *1*

(b) Find the size of the angle that the line $x-2y+3=0$ makes with the x axis (to the nearest minute). *1*

(c) For a particular curve the gradient function is $2x-4x^3$. If this curve passes through (1, 2) find its equation. 1

d) Show that the line $3x+y+7=0$ passes through the point of intersection of the lines $2x-5y-1=0$ and $4x+3y+11=0$ *2*

QUESTION 3 Start a new page *12 marks*

Marks

(a) Differentiate the following with respect to x 6

i) $\dfrac{1}{4x}$

ii) $\dfrac{x^2}{2-x}$

iii) $4x(1-2x)^4$

(b) Given $f(x)=x^4-4x^2+8$ 3

i) find $f''(-1)$

ii) find the values of x such that $f''(x)=0$

(c) Evaluate: $\int_2^4 \left(4x+x^2\right) dx$ 2

(d) $\int (1-3x)^5 \, dx =$ 1

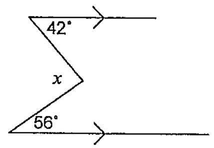
QUESTION 4 **Start a new page** *12 marks*

Marks

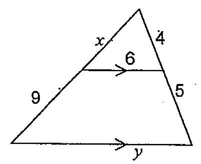
(a) Find the size of each angle of a regular octagon. **2**

(b) Find x, y **2**

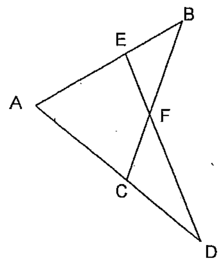
i)



ii)



(c) $AB = BC$, $\angle BAC = 75^\circ$, $\angle EFB = 45^\circ$

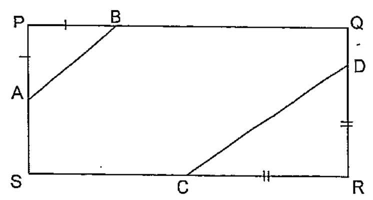


i) Copy this diagram and mark all given information **1**

ii) Find $\angle ABC$ **1**

iii) Show that $ED = AD$ giving all reasons **1**

(d) PQRS is a rectangle.
$AP = PB$, $RD = RC$



i) Copy this diagram and mark in all given information **1**

ii) Prove $\Delta PBA \,|||\, \Delta RDC$ **2**

iii) On <u>your</u> diagram produce BA and RS to meet at Point T. **1**

iv) Hence or otherwise give reasons why $AB \parallel CD$. **1**

QUESTION 5 **Start a new page** *12 marks*

Marks

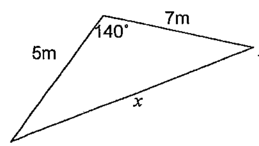
(a) Solve for $0^\circ < \theta < 360^\circ$ (correct to the nearest degree)

i) $\sin\theta = -\frac{1}{\sqrt{2}}$ **1**

ii) $\cos\theta = \frac{1}{4}\sin\theta$ **2**

(b) Show that $\frac{3}{\sec\theta - 1} - \frac{3}{\sec\theta + 1} = 6\cot^2\theta$ **2**

(c) Find the value of x to 2 decimal places. **2**



NOT TO SCALE

(d) The first term of an arithmetic sequence is 4 and the nineteenth term is 31.
Find the thirtieth term. **2**

(e) Camille is employed as a trainee technician and signs an agreement that she will be paid \$1000 for the first month, then increase by $\frac{1}{4}$% each successive month for the term of her employment. How much will she expect to earn in this job over a 20 year period? (to the nearest dollar) **3**

QUESTION 6 **Start a new page** *12 marks*

Marks

(a) i) Find the value of k if the equation $3kx^2+4x+2=0$ has equal roots *1*

ii) Find this root. *1*

(b) The roots of $4x^2+x-3=0$ are α and β. Find the value of;

i) $2\alpha+2\beta$ *1*

ii) $3\alpha\beta$ *1*

iii) $(\alpha+4)(\beta+4)$ *1*

(c) Form an equation whose roots are $\frac{-2}{3}$ and $\frac{3}{5}$ *1*

(d) Consider the two points A(1, 0), B(5, 0) and the point $P(x, y)$. *3*

i) If $\angle APB = 90^\circ$, show that the locus of P has equation $x^2-6x+y^2+5=0$

ii) Show that the locus of P is a circle and find its centre and radius.

(e) i) Sketch the parabola $16y=x^2-4x-44$ showing vertex, focus and directrix. *3*

ii) Find the equation of the tangent to the parabola at the point P(6, -2).

QUESTION 7 **Start a new page** *12 marks*

Marks

(a) Consider the curve $y=4x^3-x^4$. *6*

i) show that there are stationary points at $x=3$ and $x=0$ and determine their nature

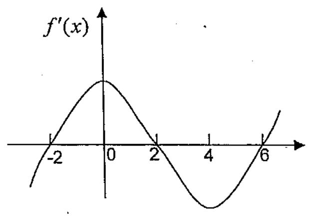
ii) find any points of inflexion

iii) sketch the curve for $-1\le x\le 4$

iv) In the domain $0\le x\le 4$, where is the curve concave down.

(b) The gradient function $y=f'(x)$ is shown in the diagram *2*

Draw a neat sketch of $y=f(x)$, given $f(0)=0$

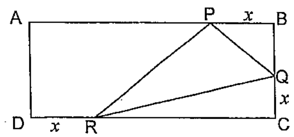


(c) ABCD is a rectangle in which AB = 12cm & AD = 8cm. The points P, Q and R lie on the perimeter of the rectangle so that PB = QC = DR = x cm. *4*

i) Copy the figure showing all dimensions.

ii) Show that the area of ΔPQR is given by $A=x^2-10x+48$

iii) Using the equation in (ii) find the least value of the area of the triangle.



Solutions to 2009 Yr 12 Mathematics Half Yearly

Question 1

a) $\frac{x}{x^2-4} + \frac{2}{x-2}$

$\frac{x + 2(x+2)}{(x-2)(x+2)}$

$= \frac{3x+4}{x^2-4}$

b) $|x-5| = 3$

$x = 8, 2$

c) $1 \cdot 33$

d) $x^2 - 5x = 0$

$x(x-5) = 0$

$x = 0, 5$

e) $(3 - \sqrt{2})^2$

$= 9 - 6\sqrt{2} + 2$

$= 11 - 6\sqrt{2}$

$\therefore a = 11, \ b = 6$

f) i) $\lim_{x \to 5} \frac{(x-5)(x^2+5x+25)}{(x-5)}$

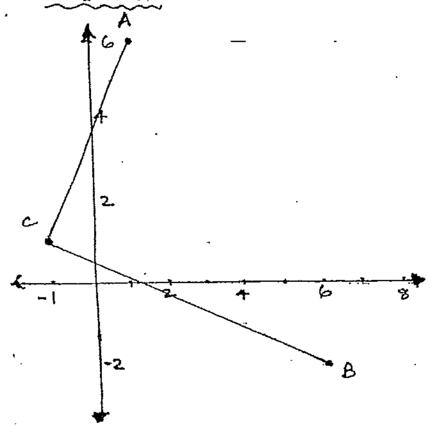
$= 5^2 + 25 + 25$

$= 75$

ii) $\lim_{x \to \infty} \frac{x^2+6x-4}{x^3-5}$

$\lim_{x \to \infty} \frac{1}{x} = 0$

Question 2



Question 2 (continued)

a) i) $BC^2 = 3^2 + 7^2$

$= 9 + 49$

$BC = \sqrt{58}$ units

ii) m of $BC = -\frac{3}{7}$

equ^n: $y - 1 = -\frac{3}{7}(x-1)$

$y = -\frac{3}{7}x + \frac{4}{7}$

OR $3x + 7y - 4 = 0$

iii) $d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right|$ $\quad a = 3, \ b = 7, \ c = -4, \ x_1 = 1, \ y_1 = 6$

$= \left| \frac{3 \times 1 + 7 \times 6 - 4}{\sqrt{58}} \right|$

$= \frac{41}{\sqrt{58}}$ units

iv) Area $= \frac{1}{2} \times \sqrt{58} \times \frac{41}{\sqrt{58}}$

$= 20 \cdot 5 \ u^2$

v) $D = (8, 3)$

b) $2y = x + 3$

$y = \frac{x}{2} + 1\frac{1}{2}$ $\quad \therefore m = \frac{1}{2}$

$\therefore \tan\theta = \frac{1}{2}$, $\theta = 26°34'$

c) $f'(x) = 2x - 4x^3$

$f(x) = x^2 - x^4 + c$

At $(1, 2)$: $\quad 2 = 1 - 1 + c$

$\therefore c = 2$

$\therefore f(x) = x^2 - x^4 + 2$

d) $2x - 5y - 1 = 0 \ \ldots (1)$

$4x + 3y + 11 = 0 \ \ldots (2)$

$4x - 10y - 2 = 0 \ \ldots (1) \times 2$

(2) − (1) $\quad 13y + 13 = 0$

$y = -1$,

$2x + 5 - 1 = 0$, $x = -2$

$\therefore$ pt of intersection $= (-2, -1)$

sub into $3x + y + 7 = 0$

$3 \times -2 - 1 + 7 = 0$

$0 = 0$

$\therefore 3x + y + 7 = 0$ passes through $(-2, -1)$

Question 3

a) i) $\frac{dy}{dx} = -\frac{1}{4}x^{-2}$

$= -\frac{1}{4x^2}$

ii) $y = \frac{x^2}{2-x}$ $\quad \frac{u}{v}$

$y' = \frac{v\,du - u\,dv}{v^2}$

$= \frac{(2-x) \times 2x - x^2 \times -1}{(2-x)^2}$

$= \frac{4x - 2x^2 + x^2}{(2-x)^2}$

$= \frac{4x - x^2}{(2-x)^2}$

$= \frac{x(4-x)}{(2-x)^2}$

iii) $y = 4x(1-2x)^4$ $\quad$ (u, v)

$y' = v\,du + u\,dv$

$= (1-2x)^4 \times 4 + 4x \times 4(1-2x)^3 \times -2$

$= 4(1-2x)^4 - 32x(1-2x)^3$

$= 4(1-2x)^3 [(1-2x) - 8x]$

$= 4(1-2x)^3 [1 - 10x]$

b) i) $f'(x) = 4x^3 - 8x$

$f''(x) = 12x^2 - 8$

$f''(-1) = 4$

ii) $12x^2 - 8 = 0$

$3x^2 - 2 = 0$

$x = \pm\sqrt{\frac{2}{3}}$

c) $\int_2^4 (4x + x^2)\,dx$

$= \left[\frac{4x^2}{2} + \frac{x^3}{3} \right]_2^4$

$= \left(2 \times 4^2 + \frac{4^3}{3}\right) - \left(2 \times 2^2 + \frac{2^3}{3}\right)$

$= 53\frac{1}{3} - 10\frac{2}{3}$

$= 42\frac{2}{3}$

d) $\int (1-3x^2)^5\,dx = \frac{(1-3x)^6}{6 \times -3} + c$

$= \frac{(1-3x)^6}{-18} + c$

Question 4

a) $S = \frac{(8-2) \times 180°}{8}$

$= \frac{1080}{8}$

$= 135°$

b) $x = 42° + 56°$

$x = 98°$

ii) $\frac{x}{9} = \frac{4}{5}$

$5x = 36$

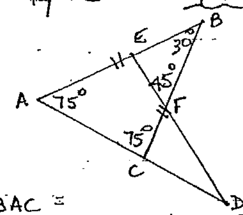
$x = 7\frac{1}{5}$

$\frac{y}{6} = \frac{9}{4}$

$4y = 54$

$y = 13\frac{1}{2}$

c) i)



c) ii) $\angle BAC = \angle BCD = 75°$ ($\angle$'s opp. equal sides in $\triangle ABC$)

$\therefore \angle ABC + 75° + 75° = 180°$ ($\angle$ sum $\triangle$)

$\angle ABC = 30°$

iii) Need to show $\triangle AED$ is isoscele

$\angle AED = 45° + 30°$ (exterior $\angle$ of $\triangle EBF$)

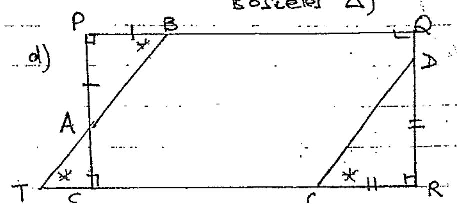
$= 75°$

Now since $\angle AED = \angle EAD = 75°$

$\therefore \triangle AED$ is isosceles

$\therefore AD = ED$ (equal sides of isosceles $\triangle$)

d)



Question 4 (continued)

I) ii) $\angle P = \angle R = 90°$ ($\angle$'s of rectangle)

given PA=PB and RD = RC

then $\frac{PB}{RC} = \frac{PA}{DR}$

$\therefore \triangle PBA \ ||| \ \triangle RDC$

(2 sides in same ratio and included angle equal)

Now $\angle PBA = \angle BTS$ (alternate $\angle$'s in $\parallel$ lines PQ + RS) – opp. sides of rectangle are $\parallel$)

also $\angle DCR = \angle PBA$ – corresponding $\angle$'s of similar $\triangle$'s

since $\angle BTS = \angle DCR$ then

$AB \parallel DC$ as corresponding $\angle$'s are equal.

Question 5

a) i) $\sin\theta = -\frac{1}{\sqrt{2}}$

$\theta = 225°, 315°$

ii) $\cos\theta = \frac{1}{4}\sin\theta$

$1 = \frac{1}{4}\tan\theta$

$\tan\theta = 4$

$\theta = 76°; 256°$

b) $LHS = \frac{3}{\sec\theta - 1} - \frac{3}{\sec\theta + 1}$

$= \frac{3(\sec\theta+1) - 3(\sec\theta - 1)}{(\sec^2\theta - 1)}$

$= 3\sec\theta + 3 - 3\sec\theta + 3$

$= \frac{6}{\tan^2\theta}$

$= 6\cot^2\theta = RHS.$

c) $x^2 = 5^2 + 7^2 - 2\times5\times7\times\cos 140°$

$= 127.623...$

$x = 11.30m$ (2 d.p.)

d) $T_1 = a = 4$

$T_{19} = a + 18d = 31$

$4 + 18d = 31$

$18d = 27$

$d = \frac{27}{18} = 1\frac{1}{2}$

$T_{30} = a + 29d$

$= 4 + 29 \times 1\frac{1}{2}$

$= 47\frac{1}{2}$

e) A G.P where

$a = \$1000$, $r = 1.0025$, $n = 240$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$= \frac{\$1000(1.0025^{240} - 1)}{1.0025 - 1}$

$= \$328\,302$

Question 6

i) $\Delta = b^2 - 4ac = 0$

$4^2 - 4\times3k\times2 = 0$

$24k = 16$, $k = \frac{2}{3}$

ii) $3\times\frac{2}{3}\ x^2 + 4x + 2$

$2x^2 + 4x + 2$

$2(x+1)^2$ $\therefore$ root $= -1$

b) i) $2(\alpha+\beta) = -2b/a = -\frac{2}{4} = -\frac{1}{2}$

ii) $3\alpha\beta = 3\times\frac{c}{a} = 3\cdot\frac{-3}{4} = -\frac{9}{4}$

iii) $(\alpha+4)(\beta+4)$

$= \alpha\beta + 4(\alpha+\beta) + 16$

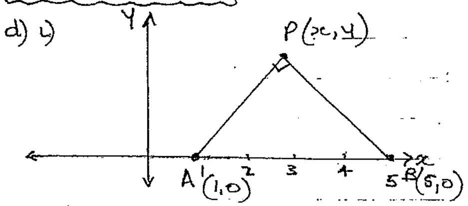
$= -\frac{3}{4} + 4\times -\frac{1}{4} + 16 = 14\frac{1}{4}$

c) $y = (x + \frac{2}{3})(x - \frac{3}{5})$

$y = (3x-2)(5x-3)$

$y = 15x^2 + x - 6$

d) i)



Question 6 continued

By Pythagoras:

$PA^2 + PB^2 = 4^2$

$(x-1)^2 + y^2 + (x-5)^2 + y^2 = 16$

$x^2 - 2x + 1 + 2y^2 + x^2 - 10x + 25 = 16$

$2x^2 + 2y^2 - 12x = -10$

$x^2 + y^2 - 6x = -5$

ii) $(x-3)^2 + y^2 = 4$

$\therefore$ centre $(3,0)$, radius $= 2$ units

e) i) $16y = x^2 - 4x - 44$

$16y = (x-2)^2 - 48$

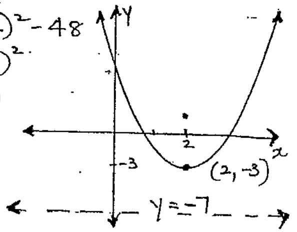
$16(y+3) = (x-2)^2$

$\therefore$ Vertex $= (2,-3)$

$a = 4$

focus $= (2,1)$

directrix: $y = -7$



ii) $y = \frac{x^2 - 4x - 44}{16}$

$y' = \frac{2x-4}{16} = \frac{x-2}{8}$

At $x = 6$, m of tangent $= \frac{6-2}{8} = \frac{1}{2}$

$\therefore$ eqn: $y + 2 = \frac{1}{2}(x-6)$

$y = \frac{x}{2} - 5$

Question 7

$\frac{dy}{dx} = 12x^2 - 4x^3 = 0$

$4x^2(3-x) = 0$

$x = 3, 0$

At $x=3$, $y=0$; $x=0$, $y=0$

$\frac{d^2y}{dx^2} = 24x - 12x^2$

At $x = 3$, $\frac{d^2y}{dx^2} < 0$ $\therefore$ max at $(3,27)$

$x = 0$, $\frac{d^2y}{dx^2} = 0$ $\therefore$ h. inflex at $(0,0)$

) $\frac{d^2y}{dx^2} = 24x - 12x^2 = 0$ for inflex. pts

$12x(2-x) = 0$

$\therefore x = 2, 0$

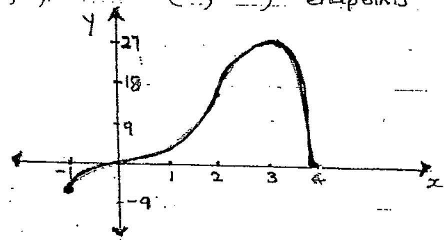
test concavity

x	-1	0	1	2	3
$\frac{d^2y}{dx^2}$	–	0	+	0	–

$\therefore$ inflex pt at $(2,16)$ + $(0,0)$

7 a) iii)

$(-1,-5)$ and $(4, 0)$ = endpoints



iv) concave down when:

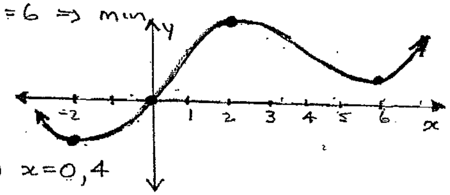
$-1 \le x < 0$

and $2 < x \le 4$

b) $x = -2 \Rightarrow$ min

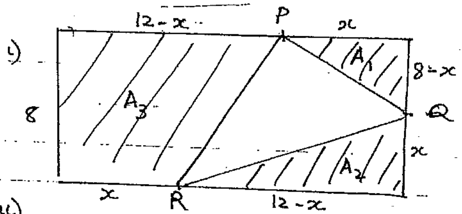
$x = 2 \Rightarrow$ max

$x = 6 \Rightarrow$ min



ii) $x = 0, 4$

c) i)



ii)

Area of $\triangle PQR$ = 12×8 – shaded areas

$A = 96 - \frac{1}{2}x\times x(8-x) - \frac{1}{2}x\times x(12-x)$ (A_1, A_2)

$-\frac{8}{2}(12 - x + x)$ (A_3 – trapezium)

$= 96 - 4x + \frac{1}{2}x^2 - 6x + \frac{1}{2}x^2 + 48$

$A = 48 - 10x + x^2$

iii) $\frac{dA}{dx} = -10 + 2x = 0$

$x = 5$

$\frac{d^2A}{dx^2} = 2 > 0$ $\therefore$ min at $x = 5$

$\therefore$ Least $A = 48 - 10\times5 + 25$

min $A = 23cm^2$