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Mrs Hickey
Mrs Gibson
Mrs Quarles
Ms Slade
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Name:______________________________

Teacher's Name:____________________

PYMBLE LADIES' COLLEGE

YEAR 12

TRIAL HIGHER SCHOOL CERTIFICATE - 1997

3/4 UNIT MATHEMATICS

Time Allowed: 2 hours

plus 5 minutes reading time

DIRECTIONS TO CANDIDATES

* Attempt all questions
* All questions are of equal value
* All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
* Standard integrals are attached.
* Approved calculators may be used.
* These are seven (7) questions in this paper.

QUESTION 1 **Marks**

(a) Evaluate

$$\int_2^4 \frac{dx}{\sqrt{16 - x^2}}$$, giving your answer in exact form **2**

(b) The polynomial $P(x) = ax^3 + bx^2 - 8x + 3$ has a factor $(x - 1)$ When divided by $(x + 2)$, the remainder is 15 **3**

Find the value of a and b

(c) Differentiate $y = \tan^{-1}(\cos x)$ with respect to x **1**

(d) Solve the inequality **3**

$$\frac{4 - x}{x} \leq 1$$

(e) Find the acute angle between the lines $y = \frac{x}{2}$ and $x + \sqrt{3}\,y + 1 = 0$ **3**

Give your answer in radians correct to two decimal places.

QUESTION 2 **(Start a new page)** **Marks**

(a) (i) Prove that **4**

$$\frac{\sin 2x}{1 - \cos 2x} = \cot x$$

(ii) Hence or otherwise obtain a value for $\cot 67\frac{1}{2}°$ in simplest surd form

(b) A(10, 1), P(8, 5) and B are points on the number plane. Point P divides the interval AB externally in the ratio 2 : 3, find the co-ordinates of B **2**

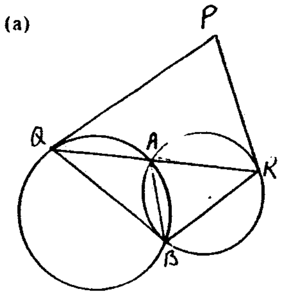
(c) (i) Find $\int \frac{6x - 1}{x^2 + 9}\,dx$ **6**

(ii) Use the substitution $u = 1 - 2x$ to evaluate

$$\int_0^{\frac{1}{2}} 2x(1 - 2x)^4\,dx$$

QUESTION 3 **(Start a new page)** **Marks**

(a)



Two circles intersect at points A and B **3**
PQ and PR are tangents and
QAR is a straight line

Prove that the points
P, Q, B, R are concyclic

(b) (i) In how many ways can 3 consonants and 2 vowels (i.e. a, e, i, o, u) be chosen from the word LOGARITHMS? **4**

(ii) What is the probability that an L will be included in the 5 letters chosen?

(c) (i) State the domain of $y = x + 3\ln x - 6$ **5**

(ii) Taking $x = 3$ as the first approximation,

Use Newton's method to find a second approximation to the root of

$$x + 3\ln x - 6 = 0$$

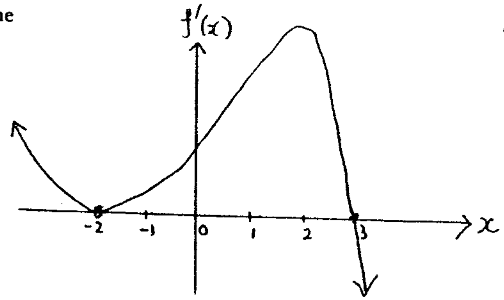
giving your answer to 2 decimal places

(iii) Explain why $x = 6 - 3\ln x$ has only one possible root

QUESTION 4 (Start a new page)

Marks

(a) The diagram shows the graph of $y = f'(x)$ 5



(i) Write down the x-coordinate(s) of the stationary point(s) of $y = f(x)$

(ii) Determine the nature of these stationary point(s) with reasons

(iii) Sketch a possible graph of $y = f(x)$

(b) Prove by Mathematical Induction that 4

$3^{2n+2} - 8n - 9$ is divisible by 64 for all positive integer n

(c) The area A cm^2 of the image of a rocket on a radar screen is given by the formula $A = \frac{12}{r^2}$ where r km is the distance of the rocket from the screen The rocket is moving away at 0 5 km/s 3

Determine the rate at which the area of the image is changing when the rocket is 10km away

QUESTION 5 (Start a new page)

Marks

(a) Evaluate $\lim_{x \to 0} \frac{\sin 3x}{2x}$ 1

(b) The rate of change of the population of a town is given by 6

$$\frac{dP}{dt} = k(P - A) \quad \text{where k and A are constants}$$

(i) Show that $P = A + Be^{kt}$ is a solution to the equation

(ii) The growth rate of a town is 2% The population was 5000 in 1980 and 6000 in 1985 Find

(a) The expected population in 1997

(b) The year in which the population is expected to reach 10000

(c) A particle is moving along the x-axis so that its acceleration after t seconds is given by 5

$$\frac{d^2x}{dt^2} = 4x(x^2 - 2)$$

The particle starts at the origin with an initial velocity of $\sqrt{6}$ cm/sec

(i) If v is the velocity of the particle, find v^2 as a function of x

(ii) Prove that the particle remains at all times within the interval

$-1 \le x \le 1$

QUESTION 6 **(Start a new page)** **Marks**

(a) $\displaystyle\int_0^{\frac{\pi}{4}} \sin^3 x \cos x \, dx$ 2

(b) (i) Differentiate $x e^{2x}$ 3

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^1 x e^{2x} dx$

(c) Tidal flow in a harbour is assumed to be simple harmonic motion and the water depth x metres at time t hours is given by 7

$$x = 20 + A \cos(nt + \alpha)$$

where A, n and α are positive constants

The depth of water is 12m at low tide and 28m at high tide which occurs 7 hours later

(i) Evaluate A and n

(ii) On a day when low tide occurs at 2 00a m find the first time period during which the water level is greater than 22m

QUESTION 7 **(Start a new page)** **Marks**

(a) Given $f(x) = \cos^{-1}\frac{x}{2} + \pi$ 7

(i) State the domain and range of this function

(ii) Find $f'(0)$

(iii) Find the inverse function $f^{-1}(x)$

(iv) Sketch both functions on the same diagram, using the same scale on both axes Label both graphs clearly

(v) State the gradient of the inverse function at the point where it crosses the x-axis

(b) A_n and B_n are two series given by 5

$$A_n = 1^2 + 5^2 + 9^2 + 13^2 + \ldots\ldots + (4n-3)^2$$

$$B_n = 3^2 + 7^2 + 11^2 + 15^2 + \ldots\ldots$$

$$for \ n = 1, 2, 3, \ldots\ldots$$

(i) Find the n^{th} term of B_n

(ii) If $S_{2n} = A_n - B_n$, prove that $S_{2n} = -8n^2$

(iii) Hence evaluate

$$101^2 - 103^2 + 105^2 - 107^2 + \ldots\ldots + 1997^2 - 1999^2$$

END OF PAPER

1991

3U Trial (12 marks each)

84

Q1. (a) $\displaystyle\int_{-2}^{4}\frac{dx}{\sqrt{16-x^2}} = \left[\sin^{-1}\frac{x}{4}\right]_{-2}^{4}$ 1

$= \sin^{-1}1 - \sin^{-1}(-\tfrac{1}{2})$

$= \frac{\pi}{2} - (-\frac{\pi}{6}) = \frac{\pi}{2}+\frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$ 1

(b) $P(x) = ax^3 + bx^2 - 8x + 3$

$P(1) = a + b - 8 + 3 = 0$

$a + b = 5$ — ① 1

$P(-2) = 15$

$-8a + 4b + 16 + 3 = 15$

$-8a + 4b = -4$ 1

$2a - b = +1$ — ②

① + ②, $3a = 6$

$a = 2$

$b = 5 - a = 3$ 1

(c) $y = \tan^{-1}(\cos x)$

$\frac{dy}{dx} = \frac{1}{1+\cos^2 x}(-\sin x) = \frac{-\sin x}{1+\cos^2 x}$ 1

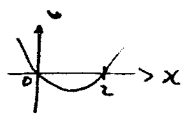
(d) $\frac{4-x}{x} \le 1$, $x \ne 0$

$(4-x)x \le x^2$

$4x - x^2 \le x^2$

$2x^2 - 4x \ge 0$

$x(x-2) \ge 0$



$x < 0$ or $x \ge 2$

(e) $y = \frac{x}{2}$, $m_1 = \frac{1}{2}$

$\sqrt{3}\,y = -x - 1$

$y = -\frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}$, $m_2 = -\frac{1}{\sqrt{3}}$ 1

$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{\frac{1}{2}+\frac{1}{\sqrt{3}}}{1-\frac{1}{2\sqrt{3}}}\right| = 1.5145\ldots$ 1

$\alpha = 0.99$ radians. 1

12

Q2. (a) (i) $LHS = \dfrac{2\sin x\cos x}{1-(\cos^2 x-\sin^2 x)}$

$= \dfrac{\cancel{2}\sin x\cos x}{\cancel{2}\sin^2 x} = \dfrac{\cos x}{\sin x} = \cot x = RHS.$

(ii) $\cot 67\tfrac{1}{2}^\circ = \dfrac{\sin 2(67\frac{1}{2}^\circ)}{1-\cos 2(67\frac{1}{2}^\circ)}$

$= \dfrac{\sin 135^\circ}{1-\cos 135^\circ} = \dfrac{\frac{1}{\sqrt{2}}}{1-(-\frac{1}{\sqrt{2}})} = \dfrac{\frac{1}{\sqrt{2}}}{\frac{\sqrt{2}+1}{\sqrt{2}}} = \dfrac{1}{\sqrt{2}+1}\cdot\dfrac{\sqrt{2}-1}{\sqrt{2}-1}$

$= \dfrac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$ // (1

(b) $8 = \dfrac{30-2x}{1}$ $\quad A(10,1) \quad B(x,y)$

$2x = 22$ (1 $\quad -2 \quad 3$

$x = 11$

$5 = \dfrac{3-2y}{1}$

$2y = -2$ (1 $\therefore B$ is $(11,-1)$ //

$y = -1$

(c) (i) $\displaystyle\int \frac{6x-1}{x^2+9}\,dx = \int\frac{6x}{x^2+9}\,dx - \int\frac{1}{x^2+9}\,dx$

$= 3\ln(x^2+9) - \frac{1}{3}\tan^{-1}\frac{x}{3} + C.$ //

(1 (1

(c) (ii) $\displaystyle I = \int_0^{\frac{1}{2}} 2x(1-2x)^4\,dx$

Let $u = 1-2x$ $\quad$ When $x=0,\ u=1$ (1

$du = -2\,dx$ $\quad x=\frac{1}{2},\ u=0$

$\displaystyle I = \int_1^0 -\left(\frac{1-u}{2}\right)(u^4)\,du$

$\displaystyle = +\frac{1}{2}\int_0^1 (u^4-u^5)\,du$

$= \frac{1}{2}\left[\frac{u^5}{5}-\frac{u^6}{6}\right]_0^1$ (1

$= \frac{1}{2}\left[\frac{1}{5}-\frac{1}{6}\right]$

$= \frac{1}{60}$ // (1

Q3 (a)

$\angle PQR = \angle ABQ$ (Angle bet tangent + chord = $\angle$ in alt. segment)
$\angle PRA = \angle ABR$ (same)

But $\angle QPR + \angle PQR + \angle PRA = 180°$ (angle sum in $\triangle PQR$).
$\therefore \angle QPR + \angle ABQ + \angle ABR = 180°$
Hence P, Q, B, R are concyclic (Opp angles are suppl.).

(b) (i) 10 letters altog. $< \begin{matrix} 3V \\ 7C \end{matrix}$

No. of ways in chosing 3 C + 2V $= {}^7C_3 \times {}^3C_2$
$= 105$

(ii) L is included.
$\therefore$ 6 C + 3 V choose 4 $< \begin{matrix} 2V \\ 2C \end{matrix}$

$$\frac{{}^6C_2 \times {}^3C_2}{105} = \frac{45}{105} = \frac{3}{7}$$

(c) (i) Domain $x > 0$.

(ii)
$y = x + 3\ln x - 6$
$y' = 1 + \frac{3}{x}$
$x_1 = 3$, $\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

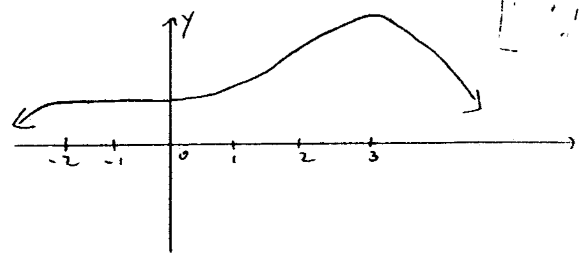
$$= 3 - \frac{3\ln 3 - 3}{2} \doteqdot 2.85$$

(iii) Fr ~~the pt~~ part (i), domain $x > 0$
$\therefore y' = 1 + \frac{3}{x} > 0$ for all x, in the given domain
Hence the curve is an increasing ~~curve~~ fn
$\therefore$ It will intersect the x-axis at one pt. only.

Q4(a) (i) x-Co-ord of stationary pts ~~are~~ are
$x = 3$ + $x = -2$

(ii) $x = 3$ Max T.P. $+ \overset{0}{\frown} -$
$x = -2$ horizontal pt of inflexion.
Grad changes from + to "0" to "+" No change of concavity.

(iii)



4(c) given $A = 12r^{-2}$, $\frac{dr}{dt} = +0.5$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt} = -\frac{24}{r^3}(+0.5) = \frac{-12}{r^3}$$

When $r = 10$ km, $\frac{dA}{dt} = \frac{-12}{10^3} = -0.012$ cm²/s

The image is shrinking at the rate of 0.012 cm²/s

4(b) next page.

Q4. (f)

Prove $3^{2n+2} - 8n - 9$ is divisible by 64

Step 1 When $n = 1$

$3^{2+2} - 8 - 9 = 81 - 17 = 64$

which is divisible by 64. (1)

Step 2. Assume that when $n = k$, $3^{2n+2} - 8n - 9$ is divisible by 64.

ie. $3^{2k+2} - 8k - 9 = 64M$ where M is an integer.

Now

$3^{2(k+1)+2} - 8(k+1) - 9$

$= 3^{2k+4} - 8k - 8 - 9$

$= 3^2(3^{2k+2}) - 8k - 17$

$= 3^2(64M + 8k + 9) - 8k - 17$

$= 3^2(64M) + 64k + 64$

$= 64(9M + k + 1)$ which is divisible by 64. (1)

Step 3. If it is true for $n = k$ it is proved true for $n = k+1$ and since s/m is true when $n = 1$

$\therefore$ it is also true for $n = 2, n = 3, n = 4 \ldots$ & so on

Hence it is true for all positive integer of n. (1)

Q5. (a) $\lim_{x \to 0} \frac{\sin 3x}{2x} = \frac{3}{2} \lim_{x \to 0} \frac{\sin 3x}{3x} = \frac{3}{2}$

(b)

(i) If $P = A + Be^{kt}$

Then $\frac{dP}{dt} = Bke^{kt} = k(Be^{kt}) = k(P - A)$

Hence $P = A + Be^{kt}$ is a sol/n to this eqn.

(ii) given $k = 0.02$

$\therefore P = A + Be^{0.02t}$.

When $t = 0$ (1980), $P = 5000$ $\therefore 5000 = A + B$

When $t = 5$ (1985), $P = 6000$ $6000 = A + B^{0.02 \times 5}$

$= A + Be^{0.1}$ (1.

Solve simultaneously, $1000 = B(e^{0.1} - 1)$

$B = \frac{1000}{e^{0.1} - 1} = 9508$

$A = -4508$

$\therefore \underline{\underline{P = -4508 + 9508\, e^{0.02t}}}$.

(α) In 1997, $t = 17$, $P = -4508 + 9508 \cdot e^{0.34}$

≈ 8850.

(β) When $P = 10000$, find t

$10000 = -4508 + 9508 \cdot e^{0.02t}$

$t = \frac{1}{0.02} \ln\left(\frac{14508}{9508}\right)$

≈ 21.13

ie. In the year 2001, the population is expected to reach 10000.

Q5 (c)(i) given $\ddot{x} = 4(x^3-2x)$

$\therefore\ v^2 = 8\int (x^3-2x)\,dx$

$= 8\left(\frac{x^4}{4} - \frac{\cancel{2}x^2}{\cancel{2}}\right) + C$

When $t=0$, $x=0$, $v^2=6$

$\therefore\ 6 = 0 + C, \quad C = 6.$

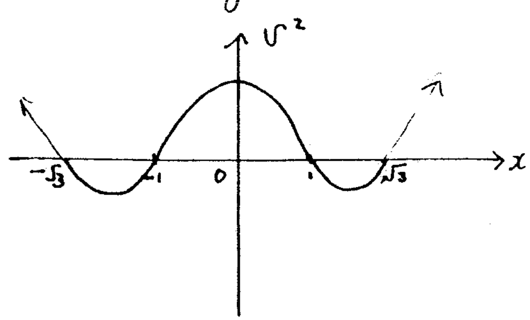
Hence $v^2 = 2x^4 - 8x^2 + 6$

$= 2(x^4 - 4x^2 + 3)$

$= 2(x^2-1)(x^2-3)$

$= 2(x+1)(x-1)(x^2-3)$

(ii) Sketch v^2 against x.



Since $v^2 \geq 0$ and $x=0$ when $t=0$

There from graph,

$-1 \leq x \leq 1$.

Q6 (a) $\int_0^{\frac{\pi}{4}} \sin^3 x \cos x \, dx$

$= \left[\frac{\sin^4 x}{4}\right]_0^{\frac{\pi}{4}}$

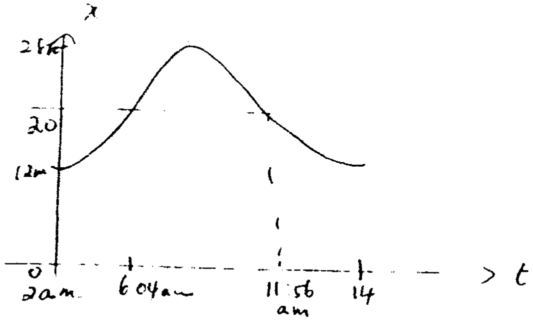
$= \frac{1}{4}\left[\left(\frac{1}{\sqrt{2}}\right)^4 - 0\right] = \frac{1}{16}$ //

(b)(i) $\frac{d}{dx}(xe^{2x}) = e^{2x} + 2xe^{2x}$

(ii) $\int_0^1 xe^{2x}\,dx = \frac{1}{2}\left[xe^{2x} - \frac{e^{2x}}{2}\right]_0^1$

$= \frac{1}{2}\left[\left(e^2 - \frac{e^2}{2}\right) - \left(-\frac{1}{2}\right)\right]$

$= \frac{1}{4}e^2 + \frac{1}{4}$ //



(i) Period $= 14 = \frac{2\pi}{n}$

$n = \frac{\pi}{7}$ (1)

amp $A = \frac{28-12}{2} = \frac{16}{2} = 8$ (1)

$\therefore \quad x = 20 + 8\cos\left(\frac{\pi}{7}t + \alpha\right)$

(ii) When $t = 0$, $x = 12$

$12 = 20 + 8\cos\left(\frac{\pi}{7}t + \alpha\right)$

$-8 = 8\cos\alpha$

$\alpha = \pi$ (1)

$\therefore \quad \underline{\underline{x = 20 + 8\cos\left(\frac{\pi}{7}t + \pi\right)}}$

When $x = 22$, find t

$22 = 20 + 8\cos\left(\frac{\pi}{7}t + \pi\right)$

$\frac{1}{4} = \cos\left(\frac{\pi}{7}t + \pi\right)$

Angles in 1st + 4th Quads (1) (1)

$\therefore \quad \frac{\pi}{7}t + \pi = 1.318 \quad \cancel{+} 2\pi - 1.318$

For positive time, 2am + (14 - 4 hr 4 min)

$\frac{\pi}{7}t + \pi = \cancel{3}\overset{\pi}{\cancel{\pi}} - 1.318$ (1)

$t =$ 4 hr 4 min later.

$\therefore$ Time is between 6:04 am and 11:56 am. 1

Q7. $f(x) = \cos^{-1}\frac{x}{2} + \pi$

(i) Domain $-2 \le x \le 2$

Range $\pi \le y \le 2\pi$.

(ii) $f'(x) = -\frac{1}{\sqrt{4-x^2}}$, $-2 < x < 2$

$f'(0) = -\frac{1}{\sqrt{4}} = \frac{-1}{2}$ // (1)

(iii) Interchange x + y

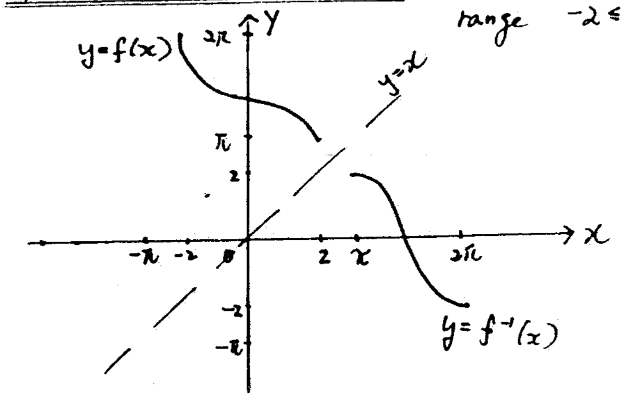
$x = \cos^{-1}\frac{y}{2} + \pi$

$\cos(x-\pi) = \frac{y}{2}$

$y = 2\cos(x-\pi)$

$\therefore \quad \underline{\underline{f^{-1}(x) = 2\cos(x-\pi)}}$, domain $\pi \le x \le 2\pi$
range $-2 \le y \le 2$.

(iv)



(v) grad of $f^{-1}(x)$ at $y = 0$ is -2 //

Q7(b) $A_n = 1^2+5^2+9^2+13^2+ \cdots +(4n-3)^2$

$B_n = 3^2+7^2+11^2+15^2+ \cdots$ for $n=1,2,3,\cdots$

(i) $B_n = (4n-1)^2$

(ii) $S_{2n} = A_n - B_n$
$= (1^2-3^2)+(5^2-7^2)+(9^2-11^2)+\cdots$
$= (1-3)(1+3)+(5-7)(5+7)+(9-11)(9+11)+\cdots$
$= -2\,[4+12+20+\cdots]$
$= -2\,[\frac{n}{2}(8+(n-1)8)]$
$= -8n^2$.

(iii) $S_{2n} = [1^2-3^2+5^2-7^2+9^2-11^2+\cdots+1997^2-1999^2]$
$- [1^2-3^2+5^2-7^2+\cdots+97^2-99^2]$

Now $4n-3 = 1997$; $4n-3=97$
$4n = 2000$ $\quad$ $4n=100$
$n = 500$ $\quad$ $n=25$

$\therefore\ S_{2n} = [-8(500)^2] - [-8(25)^2]$

$= \underline{\underline{-1995000}}$