Name:____________________

Teacher: ______________

PYMBLE LADIES' COLLEGE

MATHEMATICS

TRIAL HSC EXAMINATION

2006

Reading Time: 5 minutes
Working Time: 3 hours

Instructions to students:

- Write using blue or black biro.
- All questions may be attempted.
- Diagrams are not to scale.
- All necessary working should be shown in every question.
- Your name and your teacher's name may be written before you begin the assessment.
- Start each question in a new booklet.
- Marks may be deducted for careless or untidy work.
- Board-approved calculators may be used.
- A table of standard integrals is provided.

Question 1 (12 Marks) Use a SEPARATE writing booklet

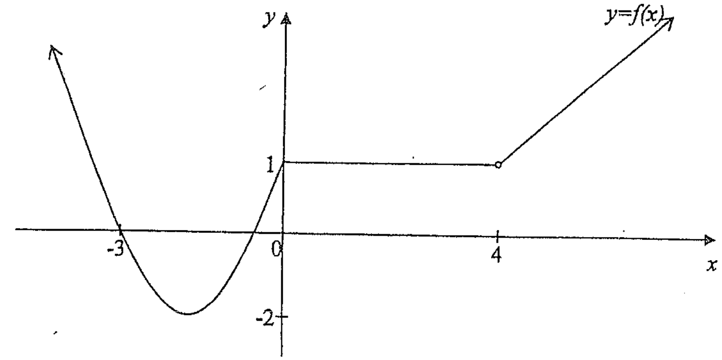
Marks

(a) Evaluate $\frac{4.26+3.81}{3\sqrt{6.27}}$. Give the answer correct to three significant figures. **2**

(b) Factorise x^3-8. **1**

(c) Find the values of a and b such that $\frac{8}{\sqrt{5}-3}=a-\sqrt{b}$. **3**

(d) The diagram below represents the graph of $y=f(x)$.



State its domain and range. **2**

(e) Find the values of x for which $|5-2x|\geq 1$. **2**

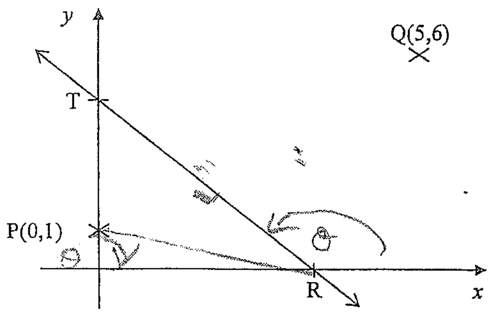
(f) Differentiate $\frac{\sqrt{x}}{4}$ with respect to x. **2**

Question 2 (12 Marks) Use a SEPARATE writing booklet

Marks

In the diagram, the points P and Q have coordinates $(0,1)$ and $(5,6)$ respectively. The line through T and R has equation $y=\dfrac{5-2x}{2}$.

Copy the diagram onto your answer booklet.



tan θ = m
tan θ = 1

(i) Find the size of the angle which the line PQ makes with the positive direction of the x-axis. **2**

(ii) Show that the equation of the line PQ is $x-y+1=0$. **1**

(iii) Given M is the point where the line PQ intersects the line RT, find the coordinates of M. **2**

(iv) Find the perpendicular distance from P to the line RT. **2**

(v) If $\angle PMR$ is a right-angle, then find the area of ΔPRM. **3**

(vi) On your diagram shade the region which satisfies $x-y+1\geq 0$, $2x+2y\leq 5$ and $y\geq 0$ simultaneously. **2**

Question 3 (12 Marks) Use a SEPARATE writing booklet

Marks

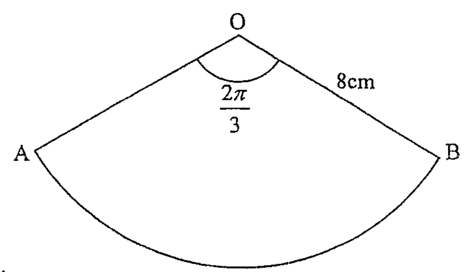
(a) Differentiate $2x\tan x$ with respect to x. **2**

(b) Given $f(x)=x^3-4x^{-1}$, find the value of $f'\left(\sqrt{2}\right)$. **2**

(c) Evaluate $\displaystyle\int_0^{\frac{\pi}{9}} \cos 3x \, dx$. **2**

(d) Find $\displaystyle\int \frac{3}{1+2x}\, dx$. **2**

(e) A cone is formed by folding the sector ABO so that the edges OA and OB coincide.



Find:

(i) the exact area of the sector ABO. **1**

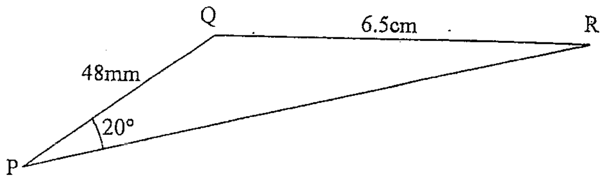
(ii) the exact length of the arc AB. **1**

(iii) the radius of the base of the cone formed. **2**

Question 4 (12 Marks) Use a SEPARATE writing booklet

Marks

(a)



PQR is a triangle with PQ = 48 mm, QR = 6.5 cm and $\angle QPR = 20°$. Find the size of $\angle PQR$ correct to the nearest degree. **3**

(b) Consider the function $f(x) = -x^5 + 15x^3$.

(i) Find the coordinates of the stationary points of the curve $y = f(x)$ and determine their nature. **4**

(ii) Sketch the curve showing all important features including the x-intercepts and points of inflexion. **4**

(iii) State the values of x for which the curve $y = f(x)$ is concave down. **1**

Question 5 (12 Marks) Use a SEPARATE writing booklet

Marks

(a) Consider the parabola $(y-4)^2 = 8(x+2)$.

(i) Write down the coordinates of the vertex. **1**

(ii) Find the focus. **2**

(iii) Find the y-intercepts. **2**

(b) If α and β are the roots of the quadratic equation $3x^2 - 4x - 1 = 0$, find the value of:

(i) $\alpha + \beta$. **1**

(ii) $\alpha\beta$. **1**

(iii) $\alpha^2 + \beta^2$. **2**

(c) Given the sequence $\ln 4$, $\ln 16$, $\ln 64$.....

(i) Show that it is arithmetic. **1**

(ii) Hence, find the sum of the first 20 terms in exact form. **2**

Question 6 (12 Marks) Use a SEPARATE writing booklet

Marks

(a) The velocity of a particle v cm/s moving in a straight line is given by $v = 1 + 2t - 3t^2$.

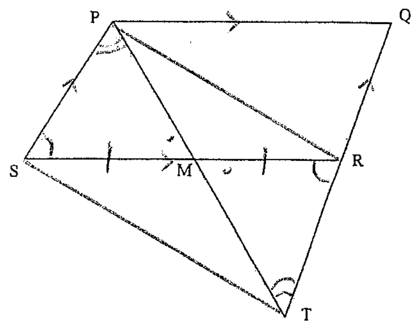
(i) If the initial displacement is 3cm to the right of 0, calculate the displacement after 2 seconds. **2**

(ii) When is the particle at rest? **2**

(iii) How far does the particle travel in the third second? **2**

(iv) Describe the motion of the particle. **1**

(b)



PQRS is a parallelogram. M is the midpoint of SR.

PM produced meets QR produced at T.

(i) Prove that $\Delta PMS \equiv \Delta TMR$. **3**

(ii) Prove that PRTS is a parallelogram. **2**

Question 7 (12 Marks) Use a SEPARATE writing booklet

Marks

(a) (i) Sketch the graph of $y = 3\sin 2x$ for $0 \le x \le 2\pi$. **2**

(ii) What is the period of the curve? **1**

(iii) State the amplitude. **1**

(iv) Find the area between the curve and the x-axis if $\frac{\pi}{4} \le x \le \frac{\pi}{2}$. **3**

(b) (i) Copy and complete the table of values for $y = 4^x$. **1**

x	-1	0	1
y			

(ii) Hence, using these three values and the trapezoidal rule, find an approximation for $\int_{-1}^{1} 4^x dx$. **2**

(iii) (α) Find the derivative of 4^x with respect to x. **1**

(β) Hence, or otherwise, find the exact value of $\int_{-1}^{1} 4^x dx$. **1**

Question 8 (12 Marks) Use a SEPARATE writing booklet

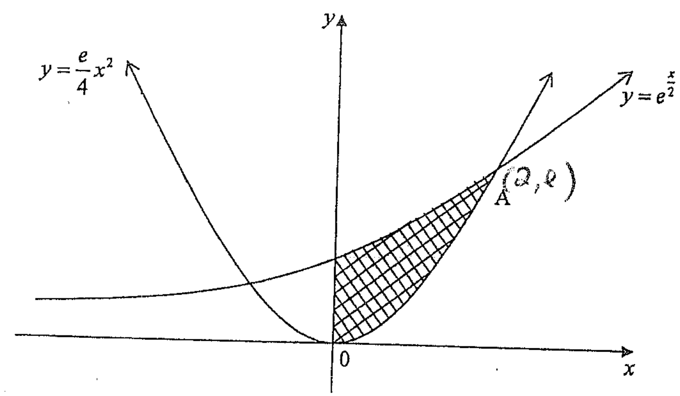
Marks

(a) Find the coordinates of the point on the curve $y = e^{3x}$ where the tangent is perpendicular to the line $y = 4 - \frac{x}{6}$. **3**

(b) (i) Show that $\sin^2 x \cos x = \cos x - \cos^3 x$. **1**

(ii) Hence, find $\frac{d}{dx}\left(\sin x - \frac{1}{3}\sin^3 x\right)$. **3**

(c)



The diagram above is of the exponential curve $y = e^{\frac{x}{2}}$ and the parabola $y = \frac{e}{4}x^2$. The point A is the first point where the two graphs meet on the right hand side of the y-axis.

(i) Show that A is the point (2,e). **2**

(ii) Show that the shaded area is $\frac{4e}{3} - 2$ units^2. **3**

Question 9 (12 Marks) Use a SEPARATE writing booklet

Marks

(a) The rate of increase of a population P(t) of people in a certain city is governed by the equation $\frac{dP}{dt} = kP$ where k is a constant and t is the time in years. The population of the city doubles every twenty years.

(i) Show that $k = \frac{1}{20}\ln 2$. **2**

(ii) In which year will the city reach a population three times that which it had at the beginning of 2006? **2**

(iii) If at the beginning of 2010 the population is 20 million, what will be the population at the beginning of the year 2060? **2**

(b) Sarah wishes to buy a car. She has worked out that she can afford repayments of \$400 a month for 5 years.

The interest rate on offer is 24% pa (reducible) calculated monthly.

Let A_n be the amount owing after n months based on a monthly repayment of \$400 and P being the amount borrowed.

(i) Give an expression for A_2. **1**

(ii) Show that $A_n = 1.02^n P - 20000(1.02^n - 1)$. **3**

(iii) Hence, determine how much money Sarah is able to borrow? **2**

Question 10 (12 Marks) Use a SEPARATE writing booklet

Marks

(a) Differentiate $\ln\left(x+\sqrt{x^2+a^2}\right)$ and hence find $\int \frac{1}{2\sqrt{x^2+a^2}}\,dx$. **3**

(b) The region bounded by the curve $y=\ln x$, the axes and the line $y=\ln 2$, is rotated about the y-axis. **3**

Find the volume of the solid formed.

(c) ABCDE is a pentagon of fixed perimeter P cm. Its shape is such that ABE is an equilateral triangle and BCDE is a rectangle.

If the length AB is x cm:

(i) Show that the length BC is $\dfrac{P-3x}{2}$ cm. **1**

(ii) Show that the area of the pentagon is given by $A=\frac{1}{4}\left[2Px-\left(6-\sqrt{3}\right)x^2\right]$. **2**

(iii) Find the value of $\dfrac{P}{x}$ for which the area of the pentagon is a maximum. **3**

End of paper

2006 HSC TRIALS – MATHEMATICS SOLUTIONS AND MARKING GUIDELINES

Q1

a) $\dfrac{4.26+3.81}{3\sqrt{6.27}}$

$= 1.07428\ldots$ 1

$= 1.07$ (3 s.f.) 1 (2)

b) $x^3 - 8$

$= (x-2)(x^2+2x+4)$ 1 R/W (1)

c) $\dfrac{8}{\sqrt{5}-3}$

$= \dfrac{8}{\sqrt{5}-3} \times \dfrac{\sqrt{5}+3}{\sqrt{5}+3}$ $\frac{1}{2}$

$= \dfrac{8(\sqrt{5}+3)}{-4}$ $\frac{1}{2}$

$= -2(\sqrt{5}+3)$

$= -6-2\sqrt{5}$ $= -6-\sqrt{20}$ $\frac{1}{2}$

$\therefore a = -6$ $\frac{1}{2}$ & $b = 20$ $\frac{1}{2}$ (3)

d) Domain : all real x ; $x \neq 4$ 1 R/W

Range : $y \geq -2$. 1 R/W (2)

e) $|5-2x| \geq 1$

$5-2x \geq 1$ OR $-5+2x \geq 1$ $\frac{1}{2}$

$-2x \geq -4$ $\frac{1}{2}$ $\quad 2x \geq 6$

$x \leq 2$ $\frac{1}{2}$ $\quad x \geq 3$ $\frac{1}{2}$ (2)

f) $\dfrac{d}{dx}\left(\dfrac{\sqrt{x}}{4}\right)$

$= \dfrac{d}{dx}\left(\frac{1}{4}x^{\frac{1}{2}}\right)$ ½

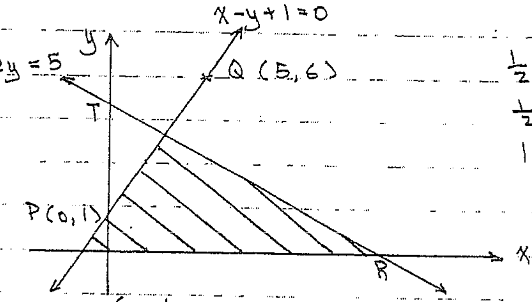
$= \frac{1}{4}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$ $\frac{1}{2}+\frac{1}{2}$

$= \frac{1}{8}x^{-\frac{1}{2}}$ ½ (2)

$= \dfrac{1}{8\sqrt{x}}$

Q2

vi) $2x+2y=5$



$\frac{1}{2}$ for labelling RT

$\frac{1}{2}$ for not colouring below

1 for correct region (2)

i) $m_{PQ} = \dfrac{6-1}{5-0} = 1$ 1

Angle line PQ with x-axis $= 45°$ 1 (2)

ii) Eq. of line PQ

$\Rightarrow 1 = \dfrac{y-1}{x-0}$ 1 R/W

$x = y-1$

$x-y+1=0$ (1)

iii) $\left[\begin{array}{l} x-y+1=0 \\ y = \dfrac{5-2x}{2} \end{array}\right.$

$x - \left(\frac{5}{2}-x\right)+1 = 0$ $\frac{1}{2}$

$2x = \frac{3}{2}$

$x = \frac{3}{4}$ $\frac{1}{2}$

$y = 1\frac{3}{4}$ $\frac{1}{2}$

$M = \left(\frac{3}{4}, 1\frac{3}{4}\right)$ $\frac{1}{2}$ (2)

iv) Eq. of RT $\Rightarrow 2y = 5-2x$

$2x+2y-5=0$ $\frac{1}{2}$

Perpendicular distance from P to RT

$= \left|\dfrac{2(0)+2(1)-5}{\sqrt{2^2+2^2}}\right|$ 1

$= \dfrac{3}{2\sqrt{2}}$ $\frac{1}{2}$ (2)

$= \dfrac{3\sqrt{2}}{4}$

v) $2x + 2y - 5 = 0$

$y = 0 \Rightarrow 2x = 5$

$x = \frac{5}{2}$ $\frac{1}{2}$

$R\left(\frac{5}{2}, 0\right)$ $\frac{1}{2}$

Length of $RM = \sqrt{\left(\frac{5}{2} - \frac{3}{4}\right)^2 + \left(0 - 1\frac{3}{4}\right)^2}$

$= 1\frac{3}{4} \times \sqrt{2}$

$= \frac{7\sqrt{2}}{4}$ $\frac{1}{2}$

$\therefore$ Area of $\triangle PRM$

$= \frac{1}{2} \times \frac{7\sqrt{2}}{4} \times \frac{3}{2\sqrt{2}}$ $\frac{1}{2}$

$= 1\frac{5}{16}$ sq. units $\frac{1}{2}$ 3

Q3

Let $u = 2x$, $\frac{du}{dx} = 2$; $v = \tan x$, $\frac{dv}{dx} = \sec^2 x$ $\frac{1}{2}$

a) $\frac{d}{dx}(2x \tan x)$

$= 2\tan x + 2x \sec^2 x$ $1\frac{1}{2}$ 2

b) $f(x) = x^3 - 4x^{-1}$

$f'(x) = 3x^2 + 4x^{-2}$ 1

$f'(\sqrt{2}) = 3(\sqrt{2})^2 + 4(\sqrt{2})^{-2}$ $\frac{1}{2}$

$= 6 + 2$

$= 8$ $\frac{1}{2}$ 2

c) $\int_0^{\frac{\pi}{9}} \cos 3x \, dx$

$= \left.\overset{\frac{1}{2}}{\frac{1}{3}} \overset{\frac{1}{2}}{\sin 3x}\right]_0^{\frac{\pi}{9}}$

$= \frac{1}{3}\sin\frac{\pi}{3} - \frac{1}{3}\sin 0$ $\frac{1}{2}$

$= \frac{\sqrt{3}}{6}$ $\frac{1}{2}$ 2

d) $\int \frac{3}{1+2x} \, dx$

$= \frac{1}{2}\,\frac{3}{2}\ln(1+2x) + C$ $\frac{1}{2}$ 1 2

e) i) Area of sector ABO

$= \frac{1}{2} \times 8^2 \times \frac{2\pi}{3}$ 1 R/W

$= \frac{64\pi}{3}$ cm^2 1

ii) Arc AB $= 8 \times \frac{2\pi}{3}$ 1 R/W

$= \frac{16\pi}{3}$ cm 1

iii) Circumference of base = length of arc AB

$2\pi r = \frac{16\pi}{3}$ 1

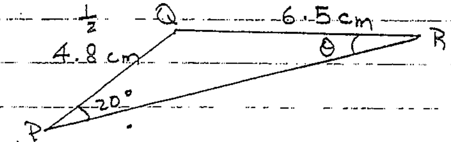
$r = \frac{8}{3}$ $\frac{1}{2}$

$\therefore$ Radius of base of cone

$= 2\frac{2}{3}$ cm. $\frac{1}{2}$ 2

Q4

a) $\frac{\sin\theta}{4.8} = \frac{\sin 20^\circ}{6.5}$ $\frac{1}{2}$



$\sin\theta = \frac{\sin 20^\circ}{6.5} \times 4.8$ $\frac{1}{2}$

$\theta = 14.629\ldots$ $\frac{1}{2}$

$\angle PQR = 180^\circ - 20^\circ - 14.629\ldots$

$= 145^\circ 22'$

$= 145^\circ$ $\frac{1}{2}$

(3)

b) $f(x) = -x^5 + 15x^3$

i) $f'(x) = -5x^4 + 45x^2 = 0$ $\frac{1}{2}$

$= -5x^2(x^2 - 9) = 0$

$\begin{cases} x = 0 \\ y = 0 \end{cases}$ $\frac{1}{2}$ $\begin{cases} x = 3 \\ y = 162 \end{cases}$ $\frac{1}{2}$ OR $\begin{cases} x = -3 \\ y = -162 \end{cases}$ $\frac{1}{2}$

$f''(x) = -20x^3 + 90x$

$f''(0) = 0$ $\frac{1}{2}$

x	-1	0	1
y''	$-$	0	$+$

$\frac{1}{2}$

$f''(3) = -20 \times 3^3 + 90 \times 3 < 0$ $\frac{1}{2}$

$f''(-3) = -20(-3)^3 + 90(-3) > 0$ $\frac{1}{2}$

$\therefore$ $(0, 0)$ is a horizontal point of inflexion

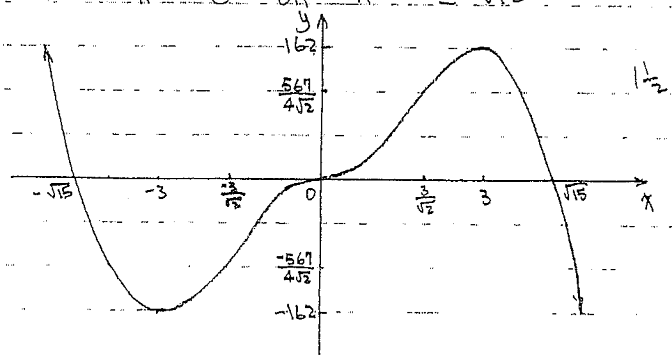
$(3, 162)$ is a max. turning point — $\frac{1}{2}$ per mistake

and $(-3, -162)$ is a min. turning point. (4)

ii) $f(x) = -x^5 + 15x^3 = 0$

$x^3(-x^2 + 15) = 0$

$x = 0$ OR $x = \pm\sqrt{15}$ $\frac{1}{2}$



$1\frac{1}{2}$

$f''(x) = -20x^3 + 90x = 0$

$-10x(2x^2 - 9) = 0$ $\frac{1}{2}$

$x = 0$, $x = \pm\frac{3}{\sqrt{2}}$ $\frac{1}{2}$

When $x = \frac{3}{\sqrt{2}}$, $y = \frac{567}{4\sqrt{2}}$, and when $x = \frac{-3}{\sqrt{2}}$, $y = \frac{-567}{4\sqrt{2}}$ $\frac{1}{2}$

x	$\frac{3}{\sqrt{2}}^-$	$\frac{3}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}^+$
$f''(x)$	$+$	0	$-$

x	$\frac{-3}{\sqrt{2}}^-$	$\frac{-3}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}^+$
$f''(x)$	$+$	0	$-$

$\frac{1}{2}$

$\therefore$ Points of inflexion at $\left(\frac{3}{\sqrt{2}}, \frac{567}{4\sqrt{2}}\right)$ and $\left(\frac{-3}{\sqrt{2}}, \frac{-567}{4\sqrt{2}}\right)$ as well. — $\frac{1}{2}$ per. (4)

iii) Concave down when $\frac{-3}{\sqrt{2}} < x < 0$ and $x > \frac{3}{\sqrt{2}}$

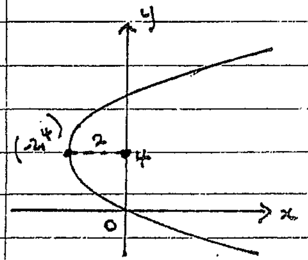
(1)

QUESTION 5

a. $(y-4)^2 = 8(x+2)$

i. (1 MARK) Vertex is $(-2, 4)$ 1 R/W

ii. (2 MARKS)



$4a = 8$

$a = 2$ 1

Focus $(0, 4)$ 1

iii. (2 MARKS) y-intercept : sub $x = 0$

$(y-4)^2 = 8(0+2)$ 1

$(y-4)^2 = 16$

$y - 4 = \pm 4$

$y = 0, 8$ $\frac{1}{2} + \frac{1}{2}$

b. $3x^2 - 4x - 1 = 0$

$a = 3, \ b = -4, \ c = -1$

i. (1 MARK) $\alpha + \beta = -\frac{b}{a}$

$= \frac{4}{3}$ 1 R/W

ii. (1 MARK) $\alpha\beta = \frac{c}{a}$

$= -\frac{1}{3}$ 1 R/W

iii. (2 MARKS) $\alpha^2 + \beta^2$

$= (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{4}{3}\right)^2 - 2 \times -\frac{1}{3}$ 1

$= \frac{22}{9}$ 1

$= 2\frac{4}{9}$

c. i. (1 MARK) $\ln 4, \ \ln 16, \ \ln 64, \ \ldots$

To prove $T_2 - T_1 = T_3 - T_2$

LHS $= T_2 - T_1$
$= \ln 16 - \ln 4$
$= \ln\left(\frac{16}{4}\right)$
$= \ln 4$

RHS $= T_3 - T_2$
$= \ln 64 - \ln 16$
$= \ln\left(\frac{64}{16}\right)$
$= \ln 4$ 1
$=$ LHS

As $T_2 - T_1 = T_3 - T_2$
then the sequence $\ln 4, \ln 16, \ln 64, \ldots$
is arithmetic.

ii. (2 MARKS) $a = \ln 4, \ d = \ln 4, \ n = 20$

$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$S_{20} = \frac{20}{2}\left[2\ln 4 + 19\ln 4\right]$ 1

$= 10 \times 21\ln 4$

$= 210 \ln 4$ $\frac{1}{2}$

$= 210 \ln 2^2$

$= 420 \ln 2$ $\frac{1}{2}$

QUESTION 6

a. $v = 1 + 2t - 3t^2$

i. (2 MARKS)

$\frac{dx}{dt} = 1 + 2t - 3t^2$

$x = t + t^2 - t^3 + c$ $\frac{1}{2}$

Sub $t = 0$, $x = 3$

$\therefore c = 3$ 1

$x = t + t^2 - t^3 + 3$

Sub $t = 2$

$x = 2 + 2^2 - 2^3 + 3$ $\frac{1}{2}$

$x = 1$

$\therefore$ Particle is 1 cm to the right of O after 2 seconds

ii. (2 MARKS) Particle is at rest when $v = 0$

i.e. $1 + 2t - 3t^2 = 0$ $\frac{1}{2}$

$3t^2 - 2t - 1 = 0$

$(3t+1)(t-1) = 0$ $\frac{1}{2}$

$t = \underset{\text{rejected as } t \geq 0}{-\frac{1}{3}}, 1$ $\frac{1}{2}$ $\frac{1}{2}$

Particle is at rest after 1 second.

iii. (2 MARKS) $x = t + t^2 - t^3 + 3$

Sub $t = 3$

$x = 3 + 3^2 - 3^3 + 3$ $\frac{1}{2}$

$= -12$ $\frac{1}{2}$

Particle has travelled $= \underset{\text{from part (i)}}{1} + 12$ $\frac{1}{2} + \frac{1}{2}$

$= 13$ cm

iv. (1 MARK) The particle starts 3 cm to the right of O. It moves to the right and stops after 1 second when it is now 4 cm to the right of O. It then moves to the left through O and carries on in that direction. 1

b. i. (3 MARKS) In ΔPMS and ΔTMR

$\angle PMS = \angle TMR$ vertically opposite angles 1

$SM = RM$ given M is the midpoint of SR

$\angle PSM = \angle MRT$ alternate angles, $PS \parallel QT$ given PQRS is a parallelogram 1

$\therefore \Delta PMS \equiv \Delta TMR$ (ASA) $\frac{1}{2}$

ii. (2 MARKS) $PS = RT$ corresponding sides of congruent Δ's 1

$PS \parallel RT$ given PQRS is a parallelogram then $PS \parallel QR$ and QR is produced to T.

Hence PRTS is a parallelogram as one pair of opposite sides are equal and parallel. 1

ALTERNATIVE METHOD

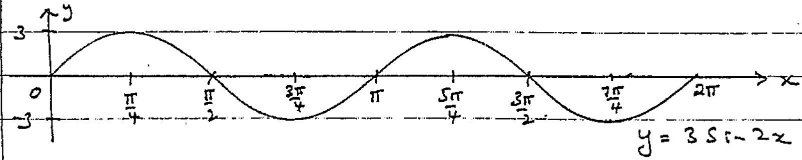
$PM = TM$ corresponding sides of congruent Δ's 1

$SM = RM$ from part (i)

Hence PRTS is a parallelogram as both diagonals PT and SR are bisected. 1

QUESTION 7

a. i. ② MARKS



$\frac{1}{2}$ mark for correct shape (no arrow-heads)

$\frac{1}{2}$ mark for correct range

$\frac{1}{2}$ mark for correct x-intercepts

$\frac{1}{2}$ mark for MAX/MIN points labelled (x-values)

ii. ① MARK

PERIOD $= \frac{2\pi}{2}$

$= \pi$ 1 R/W

iii. ① MARK

AMPLITUDE = 3 UNITS 1 R/W ignore units

iv. ③ MARKS

$\text{Area} = \int_{\pi/4}^{\pi/2} 3\sin 2x \, dx$ $\frac{1}{2}$

$= 3\int_{\pi/4}^{\pi/2} \sin 2x \, dx$

$= -\frac{3}{2}\left[\cos 2x\right]_{\pi/4}^{\pi/2}$ 1

$= -\frac{3}{2}\left\{\cos\pi - \cos\frac{\pi}{2}\right\}$ $\frac{1}{2}$

$= -\frac{3}{2}\left\{-1 - 0\right\}$ $\frac{1}{2}$

$= \frac{3}{2}$ units^2 or $1\frac{1}{2}$ units^2 ← $\frac{1}{2}$

Alternative Method: $\text{Area} = 3\int_{0}^{\pi/4} \sin 2x \, dx$

$y = 4^x$

b. i. ① MARK

x	-1	0	1
y	$1/4$	1	4

ii. ② MARKS

$\int_{-1}^{1} 4^x \, dx \doteqdot \frac{1}{2}\left(\frac{1}{4} + 4 + 2\right) = 3\frac{1}{8}$ ($\frac{1}{2}$, 1, $\frac{1}{2}$)

or $\int_{-1}^{1} 4^x \, dx \doteqdot \frac{1}{2}\left(\left(\frac{1}{4}+1\right) + (4+1)\right) = 3\frac{1}{8}$ ($\frac{1}{2}$, 1, $\frac{1}{2}$)

iii. ① MARK

α. $\frac{d}{dx} 4^x$

$= \ln 4 . 4^x$ or $4^x \ln 4$ 1.

① MARK

β. $\int_{-1}^{1} 4^x \, dx$

$= \left[\frac{4^x}{\ln 4}\right]_{-1}^{1}$ from part α. $4^x = \frac{1}{\ln 4}\frac{d}{dx}4^x$

$= \frac{1}{\ln 4}\left[4^x\right]_{-1}^{1}$

$= \frac{1}{\ln 4}\left(4 - \frac{1}{4}\right)$ 1.

$= \frac{15}{4\ln 4}$

QUESTION 8

a. (3 MARKS)

$y = e^{3x}$ ① $\qquad y = 4 - \frac{x}{6}$

$\frac{dy}{dx} = 3e^{3x} \qquad M = -\frac{1}{6} \qquad \frac{1}{2} + \frac{1}{2}$

$\therefore$ Gradient of Perpendicular $= 6$

Want $3e^{3x} = 6$ $\quad \frac{1}{2}$

$e^{3x} = 2$

$3x = \ln 2$

$x = \frac{1}{3}\ln 2$ or $\frac{\ln 2}{3}$ $\quad \frac{1}{2}$

Sub into ①

$y = e^{3 \cdot \frac{1}{3}\ln 2}$ $\quad \frac{1}{2}$

$= e^{\ln 2}$

$= 2$ $\quad \frac{1}{2}$

$\therefore$ Coordinates of required point is $(\frac{1}{3}\ln 2, 2)$

b. i. (1 MARK) To prove $\sin^2 x \cos x = \cos x - \cos^3 x$

LHS $= \sin^2 x \cos x$

$= (1 - \cos^2 x)\cos x$ $\quad 1.$

$= \cos x - \cos^3 x$

$=$ RHS

$\therefore \sin^2 x \cos x = \cos x - \cos^3 x$

ii. (3 MARKS) $\frac{d}{dx}(\sin x - \frac{1}{3}\sin^3 x)$

$= \cos x - \frac{1}{3} \cdot 3(\sin x)^2 \cdot \cos x$ $\quad \frac{1}{2} + 1\frac{1}{2}$

$= \cos x - \sin^2 x \cdot \cos x$

$= \cos x - (\cos x - \cos^3 x)$ from part (i) $\quad \frac{1}{2}$

$= \cos^3 x$ $\quad \frac{1}{2}$

c. i. (2 MARKS)

$y = e^{x/2}$

Sub $(2, e)$

$e = e^{2/2}$

$e = e^1$

True

$\therefore (2, e)$ lies on the line $y = e^{x/2}$ $\quad 1.$

$y = \frac{e}{4}x^2$

Sub $(2, e)$

$e = \frac{e}{4} \cdot 2^2$

$e = e$

True

$\therefore (2, e)$ lies on the line $y = \frac{e}{4}x^2$ $\quad 1.$

As $(2, e)$ lies on both lines and $x = 2 > 0$ (1st Quadrant) then this must be the coordinates of the point A.

ii. (3 MARKS) Shaded Area $= \int_0^2 \left(e^{x/2} - \frac{e}{4}x^2\right) dx$ $\quad 1.$

$= \left[2e^{x/2} - \frac{e}{12}x^3\right]_0^2$ $\quad 1$

$= (2e^1 - \frac{e}{12} \cdot 2^3) - (2e^0 - 0)$ $\quad \frac{1}{2}$

$= 2e - \frac{8e}{12} - 2$

or

$= 2e - \frac{2e}{3} - 2$ $\quad \frac{1}{2}$

$= e(2 - \frac{2}{3}) - 2$

$= \frac{4e}{3} - 2$ units^2 $\qquad$ Q.E.D.

QUESTION 9

a. i. $\frac{dP}{dt} = kP$

2 MARKS

$\Rightarrow \quad P = P_0 e^{kt}$ $\frac{1}{2}$

Sub $P = 2P_0$, $t = 20$

$2P_0 = P_0 e^{20k}$ 1

$e^{20k} = 2$

$20k = \ln 2$ $\frac{1}{2}$

$k = \frac{1}{20}\ln 2$

ii. $P = P_0 e^{(\frac{1}{20}\ln 2)t}$

2 MARKS

Sub $P = 3P_0$

$3P_0 = P_0 e^{(\frac{\ln 2}{20})t}$ $\frac{1}{2}$

$e^{(\frac{\ln 2}{20})t} = 3$

$\left(\frac{\ln 2}{20}\right)t = \ln 3$

$t = \frac{20\ln 3}{\ln 2}$ $\frac{1}{2}$

$\doteqdot 31.7$ years $\frac{1}{2}$

During $\left(\underset{\text{Beginning}}{2006} + \underset{\text{during}}{31}\right)$ 2037 the population will $\frac{1}{2}$

be 3 times that which it had at the beginning of 2006.

iii. $P = P_0 e^{(\frac{1}{20}\ln 2)t}$

2 MARKS

Sub $P_0 = 20$, $t = 50$

$P = 20\, e^{(\frac{\ln 2}{20})50}$ 1

$= 20\, e^{\frac{5}{2}\ln 2}$

$= 113.137085$ million 1

or 113 137 085

Refer to next page for alternative solution

(b) i. 24 % p.a. = 2 % per month

1 MARK

$A_1 = P \times 1.02 - 400$

$A_2 = A_1 \times 1.02 - 400$

$= (P \times 1.02 - 400) \times 1.02 - 400$ 1

$\therefore \quad A_2 = P \times 1.02^2 - 400(1 + 1.02)$

ii. $A_3 = A_2 \times 1.02 - 400$

3 MARKS

$= P \times 1.02^3 - 400(1 + 1.02 + 1.02^2)$ $\frac{1}{2}$

Similarly

$A_n = P \times 1.02^n - 400(\underbrace{1 + 1.02 + 1.02^2 + \cdots + 1.02^{n-1}}_{\text{This is a Geometric Series}})$ 1

$a = 1, \quad r = 1.02, \quad n = n$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$= \frac{1(1.02^n - 1)}{1.02 - 1}$ 1

$\therefore \quad A_n = P \times 1.02^n - 400\,\frac{(1.02^n - 1)}{0.02}$ $\frac{1}{2}$

$= P \times 1.02^n - 20\,000\,(1.02^n - 1)$ Q.E.D.

iii. $n = 5 \times 12$

(2) MARKS

$= 60$

$A_{60} = P \times 1.02^{60} - 20000(1.02^{60} - 1)$ $\frac{1}{2}$

But $A_{60} = 0$ [LOAN IS REPAID] $\frac{1}{2}$

$\therefore P \times 1.02^{60} - 20000(1.02^{60} - 1) = 0$

$P = \dfrac{20000(1.02^{60} - 1)}{1.02^{60}}$ $\frac{1}{2}$

$= \$13\ 904.35$ $\frac{1}{2}$

Sarah is able to borrow $13 904 to the nearest dollar.

Part (a) (iii) ALTERNATIVE SOLUTION

In 2006, population P_0

2010, population 2 Million, $\therefore t = 4$

$20 = P_0 e^{4k}$

$P_0 = \dfrac{20}{e^{(\frac{1}{20}\ln 4) \times 4}} = 17.41101127$ Million (1)

From 2006 to 2060 — 54 years. ($\frac{1}{2}$)

$\therefore P = P_0 e^{(\frac{1}{20}\ln 4) \times 54}$

$= \underline{113\ 137\ 085}$ ($\frac{1}{2}$)

QUESTION 10

a. (3) MARKS

$\frac{d}{dx} \ln(x + \sqrt{x^2 + a^2})$

$= \frac{d}{dx} \ln\left[x + (x^2 + a^2)^{1/2}\right]$

$= \dfrac{1}{x + \sqrt{x^2 + a^2}} \times \left[1 + \frac{1}{2}(x^2 + a^2)^{-1/2} \cdot 2x\right]$ $\frac{1}{2} + \frac{1}{2} +$

$= \dfrac{1}{x + \sqrt{x^2 + a^2}} \times \left[1 + \dfrac{x}{\sqrt{x^2 + a^2}}\right]$

$= \dfrac{1}{x + \sqrt{x^2 + a^2}} \times \dfrac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}$

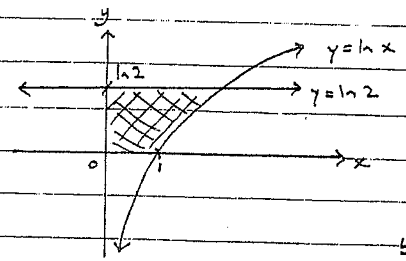
$= \dfrac{1}{\sqrt{x^2 + a^2}}$ $\frac{1}{2}$

$\int \dfrac{1}{2\sqrt{x^2 + a^2}}\, dx = \frac{1}{2} \int \dfrac{1}{\sqrt{x^2 + a^2}}\, dx$

$= \frac{1}{2} \int \left[\frac{d}{dx} \ln(x + \sqrt{x^2 + a^2})\right] dx$

$= \frac{1}{2} \ln(x + \sqrt{x^2 + a^2}) + c$ $\frac{1}{2} + \frac{1}{2}$ for $+c$

b. (3) MARKS



$y = \ln x$

$x = e^y$

$x^2 = e^{2y}$

$V = \pi \int_a^b x^2\, dy$

$= \pi \int^{\ln 2} e^{2y}\, dy$ 1

$$\therefore V = \frac{\pi}{2}\left[e^{2y}\right]_0^{\ln 2} \qquad \tfrac{1}{2}$$

$$= \frac{\pi}{2}\left\{e^{2\ln 2} - e^0\right\}$$

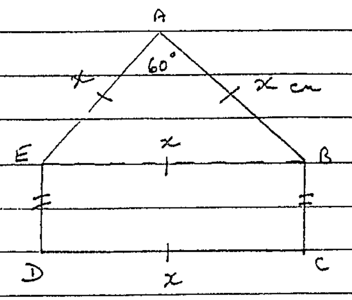
$$= \frac{\pi}{2}\left\{e^{\ln 4} - 1\right\} \qquad \tfrac{1}{2}$$

$$= \frac{\pi}{2}(4-1)$$

$$= \frac{3\pi}{2} \text{ units}^3 \qquad 1$$

[N.B. without π then 2 marks in total]

c. i. (1 MARK)



$ED = BC$ opposite sides of a rectangle BCDE

Perimeter $= AB + BC + CD + DE + EA$

$AB = AE = x = EB$ Equal sides of Equilateral triangle ABE

$EB = DC = x$ Opposite sides of rectangle BCDE

$$P = x + BC + x + BC + x$$
$$P = 3x + 2BC \qquad 1$$
$$2BC = P - 3x$$
$$BC = \frac{P-3x}{2}$$

ii. (2 MARKS)

Area = Area of ΔABE + Area of Rectangle BCDE

$$= \tfrac{1}{2}ab\sin C + LB$$

$$= \underbrace{\tfrac{1}{2}.x.x}_{\frac{1}{2}}.\underbrace{\sin 60^\circ}_{\frac{1}{2}} + x.\frac{P-3x}{2} \qquad \tfrac{1}{2}$$

$$= \frac{1}{2}x^2.\frac{\sqrt{3}}{2} + \frac{Px-3x^2}{2}$$

$$= \frac{1}{4}\left[\sqrt{3}x^2 + 2(Px-3x^2)\right]$$

$$= \frac{1}{4}\left[\sqrt{3}x^2 + 2Px - 6x^2\right] \qquad \tfrac{1}{2}$$

$$= \frac{1}{4}\left[2Px - x^2(6-\sqrt{3})\right]$$

$$\therefore A = \frac{1}{4}\left[2Px - (6-\sqrt{3})x^2\right] \qquad \text{Q.E.D.}$$

iii. (3 MARKS)

Maximum Area occurs when $\frac{dA}{dx} = 0$

$$\frac{dA}{dx} = \frac{1}{4}\left[2P - 2(6-\sqrt{3})x\right]$$

$$= \frac{1}{2}\left[P - (6-\sqrt{3})x\right] \qquad \tfrac{1}{2}$$

Hence want $\frac{1}{2}\left[P-(6-\sqrt{3})x\right] = 0$ $\qquad \tfrac{1}{2}$

$$P - (6-\sqrt{3})x = 0$$
$$P = (6-\sqrt{3})x$$
$$\boxed{\frac{P}{x} = 6-\sqrt{3}} \qquad 1$$

$$d^2A/dx^2 = -\frac{1}{2}(6-\sqrt{3}) \qquad 1.$$

$d^2A/dx^2 < 0$ for all values of x $\therefore$ Max Area occurs when $\frac{P}{x} = 6-\sqrt{3}$.