QATs
Quality Assessment Tasks

Centre Number

Student Number

2016
HIGHER SCHOOL CERTIFICATE
MID-YEAR EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11-14

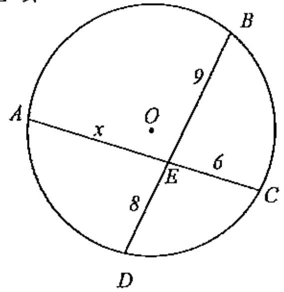
Total marks – 70

Section I Pages 2-6
10 marks
- Attempt Questions 1-10
- Allow 15 minutes for this section

Section II Pages 7-10
60 marks
- Attempt Questions 11-14
- Allow 1 hour and 45 minutes for this section

Section I (10 marks)

(1) The points A, B, C and D lie on a circle centre O as shown in the diagram below, where $AE=x$, $DE=8$, $CE=6$ and $BE=9$. 1



Which of the following is the value of x?

(A) 12

(B) 6.75

(C) 11

(D) $\frac{1}{12}$

(2) The acute angle between $x-3y=6$ and $y=3x+5$ is θ. What is the value of $\tan\theta$? 1

(A) 1

(B) $\frac{4}{3}$

(C) undefined

(D) $\frac{5}{3}$

(3) Which of the following points divides the line segment from $A(-3,2)$ to $B(5,-2)$ **externally** in the ratio 3:1? 1

(A) $(-7,4)$

(B) $(3,-1)$

(C) $(9,-4)$

(D) $(-4,7)$

(4) Which of the following is the number of nine-letter arrangements that can be formed from the letters in the word COMMITTEE? **1**

(A) 45360

(B) 60480

(C) 504

(D) 9!

(5) If $4x^2+5x-4 \equiv A(x-1)^2+B(x-1)+C$, which of the following in the value of B? **1**

(A) 5

(B) 13

(C) 1

(D) -1

(6) Which of the following is the range of $y=4\tan^{-1}\left(\frac{x}{2}\right)$? **1**

(A) $\{y: 0 \le y \le 4\pi\}$

(B) $\{y: -\pi < y < \pi\}$

(C) $\{y: -2\pi < y < 2\pi\}$

(D) $\{y: -2\pi \le y \le 2\pi\}$

(7) Which of the following equates to $\lim\limits_{(x+y)\to 0}\left\{\frac{\sin x\cos y+\cos x\sin y}{2x+2y}\right\}$? **1**

(A) $\frac{1}{x+y}$

(B) $\frac{\sin x+\sin y}{x+y}$

(C) -1

(D) $\frac{1}{2}$

(8) Given that $f(x)=\sin^{-1}(\cos x)$, which of the following is a correct statement? **1**

(A) $f'(x)=-\frac{\sin x}{\sqrt{1-x^2}}$

(B) $f'(x)=-1$

(C) $f'(x)=\frac{\cos x}{\sqrt{1-x^2}}$

(D) $f'(x)=-x$

(9) If $(x+2)$ and $(x-3)$ are factors of $P(x)=x^3-3x^2-4x+12$, which of the following is the value of the third root for $x^3-3x^2-4x+12=0$? **1**

(A) 2

(B) -84

(C) 0

(D) -8

(10) Find the values of x such that $|x+2|+|x-2|=4$? 1

(A) $\{x: x=-2,0,2\}$

(B) $\{x: x\leq -2 \text{ or } x\geq 2\}$

(C) $\{x: -2\leq x\leq 2\}$

(D) $\{x: x\geq -2\}$

Section II

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Simplify $\dfrac{\frac{a}{b}-\frac{b}{a}}{a+b}$. 2

(b) (i) Write down the identity for $\cos(A-B)$. 1

(ii) Hence show that $\cos 15^\circ = \dfrac{\sqrt{2}+\sqrt{6}}{4}$. 2

(c) When $P(x)=x^3+2x^2+x+k$ is divided by $(x-1)$ the remainder is 6. Find k. 1

(d) Solve $\dfrac{2}{x+2}<\dfrac{1}{x}$. 3

(e) Evaluate $\displaystyle\int_1^{\sqrt{10}} x\sqrt{x^2-1}\,dx$, using the substitution $u=x^2-1$. 3

(f) Let $f(x)=2x^2+3$. Use the definition $f'(t)=\lim\limits_{h\to 0}\left(\dfrac{f(t+h)-f(t)}{h}\right)$ to find the derivative of $f(x)$ at the point $x=t$. 3

Question 12 (15 marks) Use a SEPARATE writing booklet.

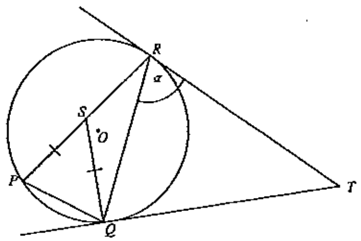
(a) Solve $\cos 2x = 2\sin^2 x,\ 0 \le x \le 2\pi$. **3**

(b) A committee of 4 people is formed from 5 men and 6 women.

(i) How many different committees could be formed. **1**

(ii) How many different committees with a majority of women could be formed. **2**

(c) In the figure below TQ and TR are tangents to a circle centre O. P lies on the circle and $QS=SP$. $\angle QRT = \alpha°$



(i) Show that $\angle RSQ = 2\alpha$. **2**

(ii) Hence show that $SQTR$ is a cyclic quadrilateral. **2**

(d) Consider the function even function $f(x) = \frac{1}{1+x^2}$.

(i) Prove that $y = 0$ is a horizontal asymptote. **1**

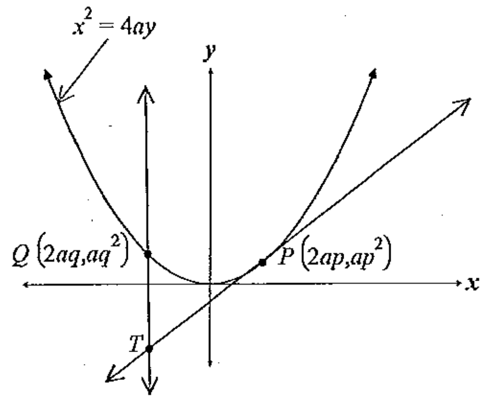
(ii) Show that $f(x) = \frac{1}{1+x^2}$ has only one stationary point, a maximum at $(0,1)$. **2**

(iii) Without finding any points of inflexion, sketch $f(x) = \frac{1}{1+x^2}$ **1**

(iv) Find an expression of $y = f^{-1}(x)$; an inverse function of $y = f(x)$ **1**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a)



In the diagram above $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. PT is a tangent at P and QT is a line parallel to the *y-axis*.

(i) Show the tangent to the parabola at P is $y = px - ap^2$. **2**

(ii) Find the coordinates of T. **1**

(iii) Find the co-ordinates of M, the midpoint of PT. **1**

(iv) Given that $pq = -1$, find the locus of M in Cartesian form. **1**

(b) It is known that a root exists for $x - 1 - \sqrt{x} = 0$, between $x = 0$ and $x = 4$. By taking $x_1 = 2$ as the first approximation to this root, use Newton's method once to find a better approximation correct to two decimal places. **2**

(c) Prove by mathematical induction that $3^n + 5$ is divisible by 8, when n is an odd number and $n \ge 1$. **3**

(d) (i) Show that $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{\pi}{4}$. **2**

(ii) Hence or otherwise find the volume of the solid of revolution when the area bounded by the graph of $y = \sin x + 1,\ 0 \le x \le \frac{\pi}{2}$ and the *x-axis* is rotated about the *x-axis*. Leave the answer in exact form. **3**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) $(x-1)$ is a factor of $P(x)=x^3-5x^2+8x-4$. Use the long division to express $P(x)$ as a product of linear factors. **2**

(ii) By graphing $P(x)=x^3-5x^2+8x-4$ and finding the relevant stationary point, find the value(s) of m so that $x^3-5x^2+8x-4-m=0$ has exactly three solutions. **2**

(b) An extension One class is made up of five girls and four boys. A class photograph is taken where all the students sit on a single long bench next to one another. In how many ways could they sit so that:

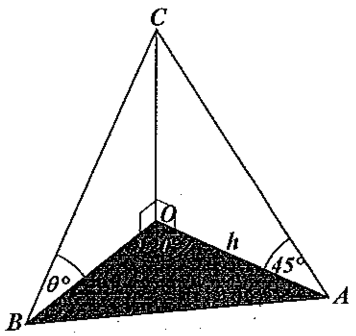
(i) The girls and boys sit in alternate seats. **1**

(ii) The girls sit together and the boys sit together. **2**

(c) (i) Show that $\dfrac{d}{dx}\left[\tan^{-1}\left(\dfrac{1}{x}\right)\right]=\dfrac{-1}{x^2+1}$. **2**

(ii) Hence prove that $\tan^{-1}x+\tan^{-1}\left(\dfrac{1}{x}\right)=c$ *(where c is a constant)* and find the value of c for $x>0$. **2**

(d)



In the above diagram, both ΔAOC and ΔBOC are right angles.
$\angle CAO=45°$, $\angle CBO=\theta°$ and $\angle AOB=120°$. $AO=h$ metres.

(i) Use the cosine rule in ΔAOB to show that $AB^2=h^2+\dfrac{h^2}{\tan^2\theta}+\dfrac{h^2}{\tan\theta}$ **3**

(ii) If $AB=\sqrt{12}$ and $h=2$, use part (i) to find the value of angle θ. **1**

END OF PAPER

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2016 HIGHER SCHOOL CERTIFICATE MID-YEAR EXAMINATION

MATHEMATICS EXTENSION 1 MARKING GUIDELINES

Section I

Multiple-choice Answer Key

Question	Answer
1	A
2	B
3	C
4	A
5	B
6	C
7	D
8	B
9	A
10	C

Section II

Question 11 (a)

Criteria	Marks
• Correct answer	2
• Achieves $\frac{a^2-b^2}{ab}\times\frac{1}{a+b}$	1

Sample answer

$$\frac{\frac{a}{b}-\frac{b}{a}}{a+b}=\frac{a^2-b^2}{ab}\times\frac{1}{a+b}$$
$$=\frac{(a-b)(a+b)}{ab}\times\frac{1}{(a+b)}$$
$$=\frac{(a-b)}{ab}$$

Question 11 (b) (i)

Criteria	Mark
• Correct answer	1

Sample answer

$\cos(A-B)=\cos A\cos B+\sin A\sin B$

Question 11 (b) (ii)

Criteria	Marks
• Correct solution	2
• Achieves $\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$	1

Sample answer

$$\cos 15^\circ=\cos(60^\circ-45^\circ)$$
$$=\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$$
$$=\frac{1+\sqrt{3}}{2\sqrt{2}}$$
$$=\frac{1+\sqrt{3}}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$
$$=\frac{\sqrt{2}+\sqrt{6}}{4}$$

Question 11 (c)

Criteria	Mark
• Correct answer	1

Sample answer

$P(1)=0$

$1+2+1+k=6$

$k=2$

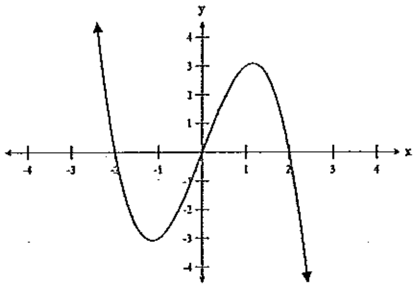
Question 11 (d)

Criteria	Marks
• Correct solution	3
• Achieves $x(x+2)(2-x)>0$	2
• Achieves $2x^2(x+2)<x(x+2)^2$	1

Sample answer

$x\neq 0,-2$

$$\frac{2}{x+2}<\frac{1}{x}$$
$$2x^2(x+2)<x(x+2)^2$$
$$x(x+2)^2-2x^2(x+2)>0$$
$$x(x+2)[x+2-2x]>0$$
$$x(x+2)(2-x)>0$$



$x<-2$ *or* $0<x<2$

Question 11 (e)

Criteria	Marks
• Correct solution.	3
• Achieves $\frac{1}{2}\int_0^9 u^{\frac{1}{2}}\,du$	2
• Achieves $u=0$ and $u=9$	1

Sample answer

$u=x^2-1$

$x=\sqrt{10}\to u=9$

$x=1 \quad \to u=0$

$u=x^2-1$

$du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$let\ I=\int_1^{\sqrt{10}} x\sqrt{x^2-1}\,dx$

$=\frac{1}{2}\int_0^9 u^{\frac{1}{2}}\,du$

$=\frac{1}{2}\times\frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^9$

$=\frac{1}{3}[27-0]$

$=9$

Question 11 (f)

Criteria	Marks
• Correct solution	3
• Achieves $f'(t)=\lim_{h\to 0}\left(\frac{2t^2+4th+2h^2+3-(2t^2+3)}{h}\right)$	2
• Writes down $f(t+h)$ and $f(t)$	1

Sample answer

$f(x)=2x^2+3$

$f(t+h)=2(t+h)^2+3$

$=2t^2+4th+2h^2+3$

$f(t)=2t^2+3$

$f'(t)=\lim_{h\to 0}\left(\frac{2t^2+4th+2h^2+3-(2t^2+3)}{h}\right)$

$=\lim_{h\to 0}\left(\frac{4th+2h^2}{h}\right)$

$=\lim_{h\to 0}(4t+2h)$

$=4t$

Question 12 (a)

Criteria	Marks
• Correct solution	3
• Achieves two correct answers	2
• Achieves $1-2\sin^2 x=2\sin^2 x$	1

Sample answer

$\cos 2x=2\sin^2 x$

$1-2\sin^2 x=2\sin^2 x$

$4\sin^2 x=1$

$\sin x=\pm\frac{1}{2}$

$x=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

Question 12 (b) (i)

Criteria	Mark
• Correct answer	1

Sample answer

$^{11}C_4 = 330$

Question 12 (b) (ii)

Criteria	Marks
• Correct answer	2
• Achieves either 110 or 15	1

Sample answer

$^6C_3 \times {}^5C_1 + {}^6C_4 = 100 + 15$

$= 115$

Question 12 (c) (i)

Criteria	Marks
• Correct proof	2
• Achieves $\angle RPQ = \alpha$ (Angle in the alternate segment)	1

Sample answer

$\angle RPQ = \alpha$ (Angle in the alternate segment)

$\angle SQP = \alpha$ (base angles of an isosceles triangle)

$\angle RSQ = 2\alpha$ (exterior angle of a triangle is equal to the two opposite interior angles)

Question 12 (c) (ii)

Criteria	Marks
• Correct proof	2
• Achieves $\angle TRQ = \angle RQT = \alpha$	1

Sample answer

$RT = TQ$ (tangents to a circle from an exterior point are equal)

$\angle TRQ = \angle RQT = \alpha$ (base angles of an isosceles triangle)

$Since\ \angle RSQ + \angle RTQ = 2\alpha + (180 - 2\alpha)$

$= 180°$

$\therefore SQTR$ is cyclic (opposite angles of a cyclic quad are supplementary)

Question 12 (d) (i)

Criteria	Mark
• Correct proof	1

Sample answer

$$\text{Horizontal asymptote} = \lim_{x\to\infty}\left(\frac{1}{1+x^2}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}+1}\right)$$

$$= \frac{0}{1}$$

$$= 0$$

Question 12 (d) (ii)

Criteria	Marks
• Correct working	2
• Achieves $f'(x) = \dfrac{-2x}{(1+x^2)^2}$	1

Sample answer

$$f(x) = \frac{1}{1+x^2}$$

$$= (1+x^2)^{-1}$$

$$f'(x) = \frac{-2x}{(1+x^2)^2}$$

Stationary points occur when $f'(x)=0$

$$\frac{-2x}{(1+x^2)^2} = 0$$

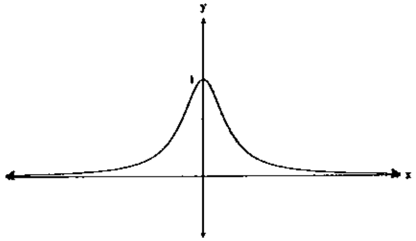
$x = 0$ $pt(0,1)$

x	-1	0	1
$f'(x)$	+	0	-

Question 12 (d) (iii)

Criteria	Mark
• Correct diagram	1

Sample answer



Question 12 (d) (iv)

Criteria	Mark
• Correct answer	1

Sample answer

$y=\frac{1}{1+x^2}$

for inverse

$x=\frac{1}{1+y^2}$

$1+y^2=\frac{1}{x}$

$y^2=\frac{1}{x}-1$

$y=\pm\sqrt{\frac{1-x}{x}}$

$\therefore f^{-1}(x)=\sqrt{\frac{1-x}{x}}$ *or* $f^{-1}(x)=-\sqrt{\frac{1-x}{x}}$

Question 13 (a) (i)

Criteria	Marks
• Correct proof	2
• Achieves $f'(2ap)=p$	1

Sample answer

$x^2=4ay$

$y=\frac{x^2}{4a}$

$y'=\frac{x}{2a}$

$y'(2ap)=p$

Equation of tangent

$y-ap^2=p(x-2ap)$

$y-ap^2=px-2ap^2$

$y=px-ap^2$

Question 13 (a) (ii)

Criteria	Mark
• Correct answer	1

Sample answer

Substitute $x=2aq$ in $y=px-ap^2$

$\therefore y=2apq-ap^2$

$\therefore T=\{2aq, 2apq-ap^2\}$

Question 13 (a) (iii)

Criteria	Mark
• Correct answer	1

Sample answer

$M=\left\{\frac{2ap+2aq}{2},\frac{2apq-ap^2+ap^2}{2}\right\}$

$=\{a(p+q), apq\}$

Question 13 (a) (iv)

Criteria	Mark
• Correct answer	1

Sample answer

$M=\{a(p+q), apq\}$

Therefore the locus of M is $y=-a$.

Question 13 (b)

Criteria	Marks
• Correct answer	2
• Correct working with some arithmetic errors	1

Sample answer

Let $f(x)=x-1-\sqrt{x}$

$f'(x)=1-\dfrac{1}{2\sqrt{x}}$

$f(2)=2-1-\sqrt{2}=-0.41421$

$f'(2)=1-\dfrac{1}{2\sqrt{2}}=0.64645$

$\therefore x_2=2+\dfrac{0.41421}{0.64645}$

$=2.6407$

$=2.64$ (2 decimal places)

Question 13 (c)

Criteria	Marks
• Correct proof	3
• Makes a substitution in step 3 (from step 2) and proves LHS=RHS	2
• Provides clear steps similar to the first three steps below	1

Sample answer

Let $f(n)=3^n+5$

Step 1: Prove the expression is true for n=1

$f(1)=3^1+5=8$ which is divisible by 8

Step 2 : Assume the expression is true for n=k, where k is an odd number

$3^k+5=8M$ *where M is an integer*

$3^k=8M-5$

Step 3 : Prove the expression is true for n=k+2

$f(k+2)=3^{k+2}+5$

$=9(3^k)+5$

$=9[8M-5]+5$ from step 2

=72M-40

$=8[9M-5]$

f(k+2) is divisible by 8

Hence if the expression is true when n=k, it is true when n=k+2
But the expression is true for n=1, ∴ it is true when n=3
If true for n=3, ∴ it is true when n=5
Therefore the expression is true for all positive $n\geq 1$ where n is odd.

Question 13 (d) (i)

Criteria	Marks
• Correct proof	1
• Achieves $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1-\cos 2x\, dx$	1

Sample answer

$$\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1-\cos 2x\, dx$$

$$= \frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left\{\left(\frac{\pi}{2}-0\right)-(0-0)\right\}$$

$$= \frac{\pi}{4}$$

Question 13 (d) (ii)

Criteria	Marks
• Correct answer	1
• Achieves $\pi\left(\frac{\pi}{4}\right)+\pi\left[-2\cos x+x\right]_0^{\frac{\pi}{2}}$	1
• Achieves $V=\pi\int_0^{\frac{\pi}{2}}(\sin x+1)^2\, dx$	1

Sample answer

$$V=\pi\int_0^{\frac{\pi}{2}}(\sin x+1)^2\, dx$$

$$=\pi\int_0^{\frac{\pi}{2}}\left(\sin^2 x+2\sin x+1\right) dx$$

$$=\pi\int_0^{\frac{\pi}{2}}\sin^2 x\, dx+\pi\int_0^{\frac{\pi}{2}} 2\sin x+1\, dx$$

$$=\pi\left(\frac{\pi}{4}\right)+\pi\left[-2\cos x+x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{\pi^2}{4}+\pi\left\{\left(0+\frac{\pi}{2}\right)-(-2+0)\right\}$$

$$=\frac{\pi^2}{4}+\frac{\pi^2}{2}+2\pi$$

$$=\left(\frac{3\pi^2}{4}+2\pi\right) units^3$$

Question 14 (a) (i)

Criteria	Marks
• Correct answer	2
• Makes a positive attempt at answer	1

Sample answer

Using log division

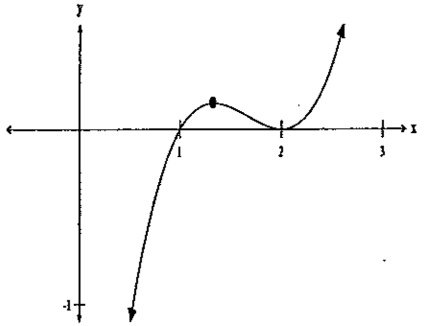
$$P(x)=(x-1)\left(x^2-4x+4\right)$$

$$=(x-1)(x-2)^2$$

Question 14 (a) (ii)

Criteria	Marks
• Correct solution	2
• Correct graph and finds relevant stationary point	1

Sample answer



Now

$P(x) = x^3 - 5x^2 + 8x - 4$

$P'(x) = 3x^2 - 10x + 8$

Stationary points occur when $P'(x)=0$

$3x^2 - 10x + 8 = 0$

$(3x-4)(x-2) = 0$

$x = \frac{4}{3}, 2$

$P\left(\frac{4}{3}\right) = \frac{4}{27}$

$x^3 - 5x^2 + 8x - 4 - m = 0$

$x^3 - 5x^2 + 8x - 4 = m$

i.e. $y = x^3 - 5x^2 + 8x - 4$ and $y = m$

$\therefore 0 < m < \frac{4}{27}$

Question 14 (b) (i)

Criteria	Mark
• Correct answer	1

Sample answer

$\textit{Number of ways} = 5! \times 4! = 2880 \textit{ ways}$

Question 14 (b) (ii)

Criteria	Marks
• Correct answer	2
• Notes $5! \times 4!$ or similar	1

Sample answer

$\textit{Number of ways} = 5! \times 4! \times 2! = 5760 \textit{ ways}$

Question 14 (c) (i)

Criteria	Marks
• Correct proof	2
• Achieves either $\left(\frac{1}{1+\frac{1}{x^2}}\right)$ or $\frac{-1}{x^2}$	1

Sample answer

$$\frac{d}{dx}\left[\tan^{-1}\left(\frac{1}{x}\right)\right] = \left(\frac{1}{1+\frac{1}{x^2}}\right) \times \frac{-1}{x^2}$$

$$= \frac{-1}{x^2+1}$$

14 (c) (ii)

Criteria	Marks
• Correct answer	2
• Correct proof or find c	1

Sample answer

Prove

$$\frac{d}{dx}\left\{\tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)\right\}=\frac{d}{dx}(c)$$

$$\frac{1}{x^2+1}+\left(\frac{-1}{x^2+1}\right)=0$$

$$0=0$$

$$\therefore \tan^{-1}x+\tan^{-1}\left(\frac{1}{x}\right)=c$$

For c substitute $x=1$

$$c=\tan^{-1}1+\tan^{-1}1$$

$$c=\frac{\pi}{2}$$

Question 14 (d) (i)

Criteria	Marks
• Correct solution	3
• Works positively by using the cosine rule	2
• Achieves $OB=\dfrac{h}{\tan\theta}$	1

Sample answer

$OC=h$

$$\tan\theta=\frac{h}{OB}$$

$$OB=\frac{h}{\tan\theta}$$

$$AB^2=h^2+\frac{h^2}{\tan^2\theta}-\left\{2\times h\times\frac{h}{\tan\theta}\times\cos 120^\circ\right\}$$

$$=h^2+\frac{h^2}{\tan^2\theta}+\left(h\times\frac{h}{\tan\theta}\right)$$

$$=h^2+\frac{h^2}{\tan^2\theta}+\frac{h^2}{\tan\theta}$$

Question 14 (d) (ii)

Criteria	Mark
• Correct answer	1

Sample answer

$$12=4+\frac{4}{\tan^2\theta}+\frac{4}{\tan\theta}$$

$$8=\frac{4}{\tan^2\theta}+\frac{4}{\tan\theta}$$

$$2=\frac{1}{\tan^2\theta}+\frac{1}{\tan\theta}$$

$$2\tan^2\theta-\tan\theta-1=0$$

$$(2\tan\theta+1)(\tan\theta-1)=0$$

$$\tan\theta=-\frac{1}{2},1$$

$$\tan\theta=1\ \ (as\,\theta<90^\circ)$$

$$\theta=45^\circ$$