Sydney Girls' High School

LABOR OMNIA VINCIT

2007
MATHEMATICS

YEAR 12
ASSESSMENT TASK 3

Time Allowed: 90 minutes

TOPICS: Quadratic Identities, Equations reducible to quadratics, Integration and the Locus (including the parabola).

Directions to Candidates

- There are five (5) questions.
- Attempt ALL questions.
- Questions are of NOT of equal value.
- Start each question on a new page.
- Write on one side of the paper only.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.
- Diagrams are NOT drawn to scale.
- Board-approved calculators may be used.

Total: 100 marks

QUESTION 1 (20 marks) — Marks

(a) Find

(i) $\int x^3 - 3x^2 + 2\ dx$ — 2

(ii) $\int \frac{7x^4 + 1}{x^2}\ dx$ — 2

(iii) $\int \frac{x^2 - 4}{x - 2}\ dx$ — 2

(iv) $\int (2 - 5x)^6 dx$ — 2

(b) State the centre and the radius of the circle $(x + 2)^2 + (y - 1)^2 = 25$ — 2

(c) Find the equation of the locus of a point which moves so that it is equidistant from the point (2, 4) and the line $y = 0$. — 2

(d) Find a, b and c if $2x^2 + 3x - 5 \equiv ax(x - 1) + bx + c$. — 4

(e) A point P(x,y) moves so that it is always equidistant from the point (-1, 2) and (3, 4). Find the equation of its path. — 4

QUESTION 2 (20 marks) Marks

(a) If $\frac{dy}{dx} = \sqrt{2x+1}$ and when $x = 4$, $y = 0$ find an expression for y 4

(b) Find the area enclosed by $y = x^2 + x$ and the x – axis. 4

(c) During an experiment the following values of x and $y = f(x)$ were recorded.

x	1.00	1.25	1.50	1.75	2.00	2.25	2.50
$f(x)$	3.43	2.17	0.38	1.87	2.65	2.31	1.97

Use the trapezoidal rule to find the approximate value of $\int_{1.00}^{2.50} f(x)\,dx$.

Correct to one decimal place. 4

(d) Find $\int_0^4 x\sqrt{x}\,dx$ 3

(e) For the parabola $y = \frac{x^2}{8} - 3$ find:

(i) the vertex
(ii) the focus 5
(iii) the equation of the directrix

QUESTION 3 (21 marks) Marks

(a) Find the equation of the parabola with focus (2, 4) and the directrix $y = -2$ 3

(b) If $\int_0^k (3 - 2x)\,dx = -4$, find the value of k, given that k is positive. 4

(c) Solve (i) $(x^2 - 2x)^2 - 4(x^2 - 2x) + 3 = 0$

(ii) $3^{2x} + 2.3^x - 15 = 0$ 4

(d) Show that the curves $y = x^2 - 2x - 3$ *and* $y = 1 - x^2$ meet at the points $(-1, 0)$ and $(2, -3)$.

Hence, find the area between the curves. 5

QUESTION 4 (19 marks)

(a) Evaluate $\int_2^4 \sqrt{16 - x^2}\,dx$ using Simpson's Rule with three function values to find an approximation. Answer correct to two decimal places. 4

(b) Calculate the area enclosed by the curve $y = x^3$, the Y-axis and the lines $y = 1$ and $y = 8$. 4

(c) Determine the area of the region enclosed by the parabola $y = x^2 + 1$ and the line $y = 10$. 5

(d) (i) Sketch the function $y = \sqrt{a^2 - x^2}$

(ii) Find the volume of the solid formed when the area enclosed by $y = \sqrt{a^2 - x^2}$ and the X-axis is rotated about the X-axis.

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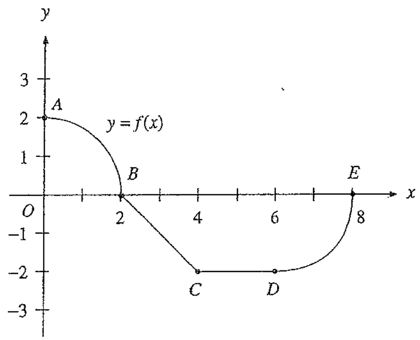
QUESTION 5 (20 marks) Marks

(a) Given A (-a, 0) and B (a, 0) find the locus of P(x, y) if AP is perpendicular to BP. 3

(b)



The graph of the function f consists of a quarter circle AB, a straight line segment BC, a horizontal straight line segment CD, and a quarter circle DE as shown above.

(i) Evaluate $\int_0^8 f(x)\ dx$ 2

(ii) For what values of x satisfying $0 < x < 8$ is the function f NOT differentiable? 1

QUESTION 5-continued (20 marks) Marks

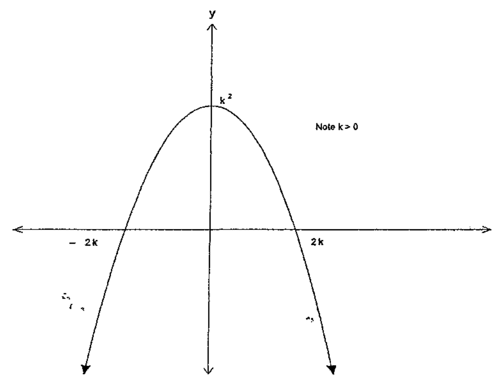
(c) Given the curves $y = x^2$ and $y = 8 - x^2$:

(i) Find the points of intersection of the curves 2

(ii) On the same set of axes, draw a neat sketch of the curves $y = x^2$ and $y = 8 - x^2$ showing their points of intersection. 1

(iii) Hence find the volume of the solid of revolution formed when the region between the curves $y = x^2$ and $y = 8 - x^2$ in the $x - y$ plane is rotated about the Y-axis. 4

(d)



(i) Find the equation of the parabola 3

(ii) If the area enclosed by the parabola and the X-axis is $\frac{343}{3}$ square units, find the value of k. 4

THE END

MATHEMATICS - 2007

YEAR 12 - ASSESSMENT TASK 3

Question 1 (20 marks)

a) i) $\int x^3 - 3x^2 + 2\,dx$ (2)

$= \frac{x^4}{4} - x^3 + 2x + C$

(ii) $\int \frac{7x^4+1}{x^2}\,dx = \int 7x^2 + x^{-2}\,dx$

$= \frac{7x^3}{3} - \frac{1}{x} + C$ (2)

(iii) $\int \frac{x^2-4}{x-2}\,dx = \int \frac{(x-2)(x+2)}{(x-2)}\,dx$

$= \int x + 2\,dx$

$= \frac{x^2}{2} + 2x + C$ (2)

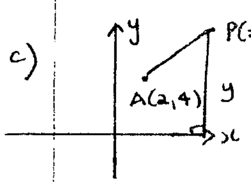
(iv) $\int (2-5x)^6\,dx$

$= \frac{(2-5x)^7}{-35} + C$ (2)

b) Centre $(-2, 1)$

radius = 5 units (2)

c)



$y = 0$ (2)

$(x-2)^2 + (y-4)^2 = y^2$

$x^2 - 4x + 4 + y^2 - 8y + 16 = y^2$

$x^2 - 4x - 8y + 20 = 0$

d) $2x^2 + 3x - 5 \equiv ax(x-1) + bx + c$

$\equiv ax^2 - ax + bx + c$

$\therefore a = 2, \quad c = -5$

$-a + b = 3$

$b = 3 + 2$

$b = 5$ (4)

$\therefore a = 2, b = 5, c = -5$

e) $(PA)^2 = (PB)^2$

$(x+1)^2 + (y-2)^2 = (x-3)^2 + (y-4)^2$

$x^2 + 2x + 1 + y^2 - 4y + 4 =$

$x^2 - 6x + 9 + y^2 - 8y + 16$

$2x - 4y + 5 = -6x - 8y + 25$

$8x + 4y - 20 = 0$ (4)

OR $2x + y - 5 = 0$

Question 2 (20 marks)

a) $\frac{dy}{dx} = \sqrt{2x+1}$

$= (2x+1)^{1/2}$ (4)

$y = \frac{2(2x+1)^{3/2}}{3 \times 2} + C$

$= \frac{(2x+1)^{3/2}}{3} + C$

at $x = 4, \; y = 0$

$0 = \frac{(8+1)^{3/2}}{3} + C$

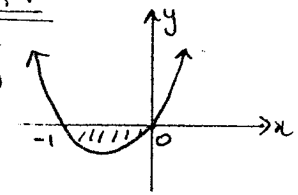
$\therefore C = -9$

$y = (2x+1)^{3/2} - 9$

Question 2 - con't

b) $y = x^2 + x$

$= x(x+1)$



$x(x+1) = 0$

$x = 0$ or $x = -1$

Area $= \left| \int_{-1}^{0} x^2 + x\,dx \right|$

$= \left| \frac{x^3}{3} + \frac{x^2}{2} \right|_{-1}^{0}$

$= \left| 0 - \left(-\frac{1}{3} + \frac{1}{2}\right) \right|$ (4)

$= \frac{1}{6}$ units^2

c)

x	1.00	1.25	1.50	1.75	2.00	2.25	2.50
$f(x)$	3.43	2.17	0.38	1.87	2.65	2.31	1.97
w	1	2	2	2	2	2	1

$h = \frac{0.25}{2} = 0.125$

$\int_a^b f(x)\,dx \doteq \frac{h}{2} \sum w f(x)$

$\doteq \frac{1}{8} \times 24.16$

$\doteq 3.0$ (4)

d) $\int_0^4 x\sqrt{x}\,dx$

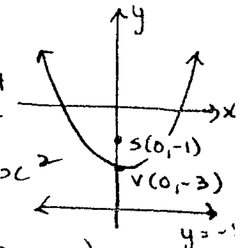
$= \int_0^4 x^{3/2}\,dx$

$= \left[\frac{2x^{5/2}}{5} \right]_0^4$

$= \left[\frac{2}{5}(4)^{5/2} - 0 \right]$ (3)

$= 12\frac{4}{5}$

e) $y = \frac{x^2}{8} - 3$



$8y = x^2 - 24$

$8y + 24 = x^2$

$8(y+3) = x^2$

(i) Vertex $(0, -3)$

(ii) $4a = 8$

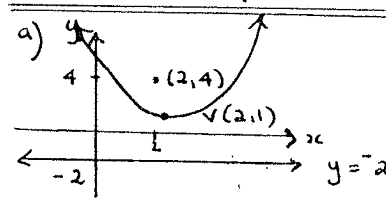
$a = 2$

Focus $(0, -1)$ (5)

(iii) $\therefore$ directrix $y = -5$

Question 3 (21 marks)

a)



Vertex $(2, 1)$ $\quad a = 3$ (3)

$(x-2)^2 = 4(3)(y-1)$

$(x-2)^2 = 12(y-1)$

b) $\int_0^k (3-2x)\,dx = -4$

$\int_0^k (3-2x)\,dx = \left[3x - x^2\right]_0^k$

$= 3k - k^2$ (4)

ie/ $3k - k^2 = -4$

$0 = k^2 - 3k - 4$

$0 = (k-4)(k+1)$

$k = 4$ or $k = -1$

only solution $k = 4$, given $k > 0$

Question 3 - con't

c) i) $(x^2 - 2x)^2 - 4(x^2 - 2x) + 3 = 0$

let $m = x^2 - 2x$

$m^2 - 4m + 3 = 0$

$(m-3)(m-1) = 0$

$m = 3$ or $m = 1$

$x^2 - 2x - 3 = 0$ $\quad$ $x^2 - 2x - 1 = 0$

$(x-3)(x+1) = 0$ $\quad$ $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

(2) $x = 3$ or $x = -1$ $\quad$ $x = \frac{2 \pm \sqrt{4+4}}{2}$

$x = \frac{2 \pm \sqrt{8}}{2}$

$x = \frac{2 \pm 2\sqrt{2}}{2}$

$x = 1 \pm \sqrt{2}$

∴ Solutions $x = -1, 3$ or $1 \pm \sqrt{2}$

(ii) $3^{2x} + 2.3^x - 15 = 0$

let $m = 3^x$

$m^2 + 2m - 15 = 0$

$(m+5)(m-3) = 0$

$m = -5$ $\quad$ $m = 3$

(2)

$3^x = 3$

$x = 1$ only solution

d) $y = x^2 - 2x - 3$ and $y = 1 - x^2$

∴ $1 - x^2 = x^2 - 2x - 3$

$0 = 2x^2 - 2x - 4$

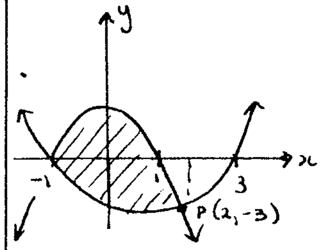
$0 = x^2 - x - 2$

$0 = (x-2)(x+1)$

$x = 2$ $\quad$ $x = -1$

$y = -3$ $\quad$ $y = 0$

$P_1(2,-3)$ and $P_2(-1,0)$



(5)

$A = \int_{-1}^{2} (1-x^2) - (x^2-2x-3)\, dx$

$= \int_{-1}^{2} 1 - x^2 - x^2 + 2x + 3\, dx$

$= \int_{-1}^{2} -2x^2 + 2x + 4\, dx$

$= \left[-\frac{2x^3}{3} + x^2 + 4x \right]_{-1}^{2}$

$= \left(-\frac{16}{3} + 4 + 8\right) - \left(\frac{2}{3} + 1 - 4\right)$

$= 6\frac{2}{3} - -2\frac{1}{3}$

$= 9$ units^2

Question 4 (19 marks)

a) $\int_2^4 \sqrt{16 - x^2}\, dx$

x	$f(x)$	w
4	0	1
3	$\sqrt{7}$	4
2	$\sqrt{12}$	1

$\int_2^4 \sqrt{16-x^2}\, dx \doteq \frac{1}{3} \sum w f(x)$

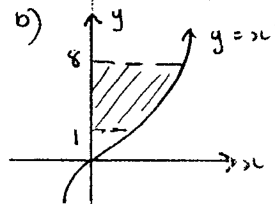
$\doteq \frac{1}{3} \times 14.047106$

$\doteq 4.68$ (2 decimal place)

(4)

Question 4 - con't

b)



$y = x^3$

$y^{1/3} = x$

$V = \int_1^8 (y^{1/3})\, dy$

$= \int_1^8 y^{1/3}\, dy$

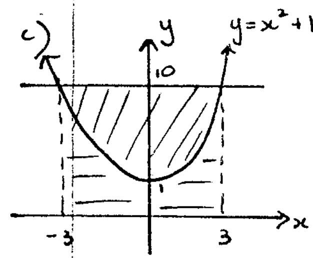
$= \frac{3}{4} \left[y^{4/3} \right]_1^8$

$= \frac{3}{4} [16 - 1]$

$= \frac{45}{4}$ units^2

(4)

c)



Points of intersection

$10 = x^2 + 1$

$x^2 = 9$

$x = \pm 3$

∴ Required Area $= 6 \times 10\ u^2 -$ Area under Parabola

Area Under Parabola

$A = \int_{-3}^{3} (x^2 + 1)\, dx$

$= 2 \int_0^3 x^2 + 1\, dx$

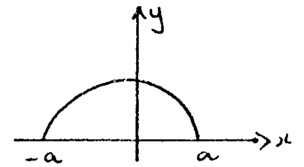
$= 2 \left[\frac{x^3}{3} + x \right]_0^3$

$= 2(12) = 24$ units^2

∴ Required Area $= 60 - 24$

$= 36$ units^2 (5)

d) (i) $y = \sqrt{a^2 - x^2}$



(ii) $V = \pi \int_{-a}^{a} \left(\sqrt{a^2 - x^2} \right)^2 dx$

$= 2\pi \int_0^a a^2 - x^2\, dx$

$= 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a$

$= 2\pi \left[a^3 - \frac{a^3}{3} \right]$

$= \frac{4\pi a^3}{3}$ units^2 (6)

Question 5 - (20 marks)

a) $AP \perp BP$

Gradient $AP = \frac{y}{x+a}$

Gradient $BP = \frac{y}{x-a}$

∴ $\left(\frac{y}{x+a}\right) \cdot \left(\frac{y}{x-a}\right) = -1$ (3)

$\frac{y^2}{x^2 - a^2} = -1$

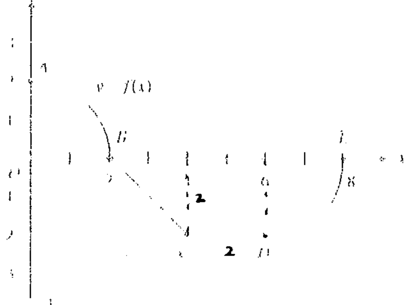
$y^2 = -x^2 + a^2$

$y^2 + x^2 = a^2$

Question 5 - con't

b) i) $\int_0^8 f(x)\,dx = -(\frac{1}{2}\times2\times2) - 2\times2$

$= -6$



Area of AB cancels Area of DE (2)

(ii) The function is NOT differentiable at any point where the curve is not smooth or not continuous. Hence, it is not differentiable at $x=2$ and $x=4$

(the end points $x=0$ and $x=8$ are not included, and at $x=6$, the gradient is continuous). (1)

c) i) $x^2 = 8 - x^2$

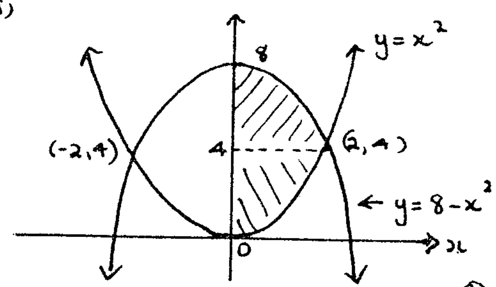
$2x^2 - 8 = 0$

$x^2 - 4 = 0$ (2)

$x = 2$ or $x = -2$

$y = 4$ $\quad$ $y = 4$

(ii)



(1)

(iii) $y = x^2$

$y = 8 - x^2$

$x^2 = 8 - y$

$V = \pi\int_a^b x^2\,dy$

$= \pi\int_4^8 (8-y)\,dy + \pi\int_0^4 y\,dy$

$= \pi\left[8y - \frac{y^2}{2}\right]_4^8 + \pi\left[\frac{y^2}{2}\right]_0^4$

$= \pi\left[(32-24) + (8)\right]$

$= 16\pi$ units^3 (4)

d) (i) Equation in the form

$(x-0)^2 = 4a(y-k^2)$

$x^2 = 4a(y-k^2)$

at $y=0$, $x = 2k$

$4k^2 = -4ak^2$

$\therefore a = -1$ (3)

$\therefore$ Equation

$x^2 = -4(y-k^2)$

$x^2 = -4y + 4k^2$

$\therefore y = k^2 - \frac{x^2}{4}$

Question 5 - con't

(ii) Area $= 2\int_0^{2k}\left(k^2 - \frac{x^2}{4}\right)dx$

$= 2\left[k^2x - \frac{x^3}{12}\right]_0^{2k}$

$= 2\left[2k^3 - \frac{8k^3}{12}\right]$

$= \frac{8k^3}{3}$

Now, Area $= \frac{343}{3}$

$\therefore \frac{343}{3} = \frac{8k^3}{3}$

$343 = 8k^3$

$k^3 = 42\,^7\!/_8$

$K = 3.5$ (4)