P Q4, Q6 Q

Radians and Trigonometric Differentiation

2-12

1 Differentiate these equations

(i) $y = 4x + \sin 2x$ (ii) $y = 2\tan \pi x$ (iii) $y = 4\cos\frac{\pi x}{2}$

(iv) $y = \frac{\sin x}{x}$ (v) $y = x\cos x$ (vi) $y = (\tan x + 1)^3$

(vii) $y = \sin 3x - \tan\frac{x}{3}$ (viii) $y = \sin x \cos x$ (ix) $y = \sin^2 x$

2 (i) Find the equation of the tangent to $y = 2\sin x$ at the point $\left(\frac{\pi}{6}, 1\right)$.

(ii) Find the equation of the normal to $y = \cos 3x$ at $\left(\frac{\pi}{2}, 0\right)$.

(iii) Find the equation of the tangent at $\left(\frac{\pi}{4}, 1\right)$ on $y = \tan x$.

3 Find (x, y) on the curve $y = \cos x$, $0 \le x \le \frac{\pi}{2}$, where the tangent is parallel to the line $2y + x = 10$.

4 Find the co-ordinates of the points on $y = \cos 2x$ in $0 \le x \le 2\pi$, where the curve has a gradient of -1.

5 Find the co-ordinates of the point on $y = \tan x$, $0 \le x \le \frac{\pi}{2}$, where the normal has a gradient of $-\frac{1}{2}$.

6 Find the x co-ordinate of the 4 stationary points on $y = \sin 2x - \sqrt{3}x$ in $0 \le x \le 2\pi$ and determine their nature.

7 (i) Find the co-ordinates of the stationary points on the curve $y = x - 2\sin x$, in $0 \le x \le 2\pi$ and determine their nature.

(ii) Hence sketch the curve $y = x - 2\sin x$, in $0 \le x \le 2\pi$.

(iii) Where on the curve $y = x - 2\sin x$, in $0 \le x \le 2\pi$ is $\frac{dy}{dx} \ge 0$?

8 Where on the curve $y = \sin x$, in $0 \le x \le 2\pi$ is

(i) $\frac{dy}{dx} > 0$

(ii) $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$

Trigonometric Integration

2-13

1 Find expressions for these indefinite integrals.

(i) $\int \cos 2x \, dx$ (ii) $\int 4\sin 3x \, dx$ (iii) $\int 8\sec^2 5x \, dx$

(iv) $\int (6x + \sin \pi x) \, dx$ (v) $\int \cos \frac{x}{2} \, dx$ (vi) $\int (x^2 + \sec^2 \frac{\pi}{2} x) \, dx$

(vii) $\int \frac{1}{2}\sin 4x \, dx$ (viii) $\int (2\sec^2 x + \cos 2x) \, dx$ (ix) $\int (1 + \sin x) \, dx$

(x) $\int (1 - \sec^2 \frac{1}{2} x) \, dx$ (xi) $\int (\pi + \cos x) \, dx$ (xii) $\int (\pi - \sin \pi x) \, dx$

2 Find the value of these definite integrals.

(i) $\int_0^{\frac{\pi}{4}} \sin x \, dx$ (ii) $\int_0^1 \sec^2 3x \, dx$ (iii) $\int_0^2 4\cos x \, dx$

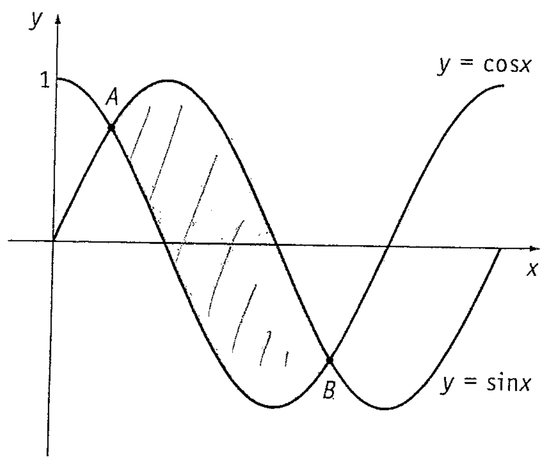
(iv) $\int_0^{\frac{\pi}{2}} (1 + \cos x) \, dx$ (v) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x) \, dx$ (vi) $\int_0^{2\pi} \sin 3x \, dx$

3 A curve has a gradient function given by $\frac{dy}{dx} = \sin 2x$. The curve passes through the point $(\pi, 3)$. Find the equation of the curve.

4 The diagram shows the graphs of $y = \sin x$ and $y = \cos x$.

(i) Write down the co-ordinates of points A and B.

(ii) Hence calculate the size of the shaded area.



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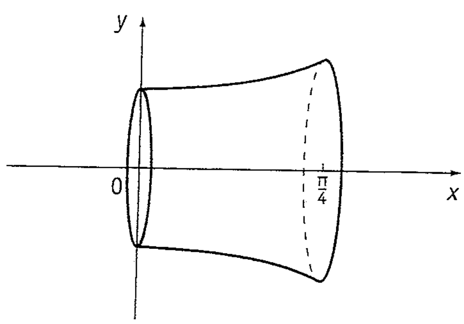


The section of the curve $y = \sec x$ which is between $x = 0$ and $x = \frac{\pi}{4}$ was rotated about the x axis, as shown in the diagram.

Calculate the volume contained in the solid.

Radians & Trig. Differentiation

1i) $y = 4x + \sin 2x$

$\therefore y' = 4 + 2\cos 2x$ ✓

ii. $y = 2\tan \pi x$

$\therefore y' = 2\pi \sec^2 \pi x$ ✓

iii. $y = 4\cos \frac{\pi x}{2}$

$\therefore y' = -2\pi \sin \frac{\pi x}{2}$ ✓

iv. $y = \frac{\sin x}{x}$

$y' = \frac{x.\cos x - \sin x}{x^2}$ ✓

v. $y = x\cos x$

$y' = \cos x + x.(-\sin x)$

$= \cos x - x\sin x$ ✓

vi. $y = (\tan x + 1)^3$

$y' = 3(\tan x + 1)^2 . \sec^2 x$

$= 3\sec^2 x(\tan x + 1)^2$ ✓

vii. $y = \sin 3x - \tan \frac{x}{3}$

$y' = 3\cos 3x - \frac{1}{3}\sec^2 \frac{x}{3}$ ✓

viii. $y = \sin x \cos x$

$y' = \cos x(\cos x) + \sin x(-\sin x)$

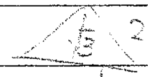
$= \cos^2 x - \sin^2 x$

$= \cos 2x$ ✓

ix. $y = \sin^2 x$

$y' = 2\sin x . \cos x$ ✓

2i. $y = 2\sin x$



$y' = 2\cos x$

when $x = \frac{\pi}{6}$, $y' = 2\cos\frac{\pi}{6} = 2.\frac{\sqrt{3}}{2} = \sqrt{3}$

$\therefore$ equ. is $y - 1 = \sqrt{3}(x - \frac{\pi}{6})$ ✓

ie. $y = \sqrt{3}x - \frac{\sqrt{3}\pi}{6} + 1$

ii. $y = \cos 3x$

$y' = -3\sin 3x$

when $x = \frac{\pi}{2}$, $y' = -3\sin\frac{3\pi}{2} = 3$

$m_1 = 3 \quad \therefore m_2 = -\frac{1}{3}$

[$m_1 m_2 = -1$ for perp. lines]

$\therefore$ equ normal is $y - 0 = -\frac{1}{3}(x - \frac{\pi}{2})$

ie. $y = -\frac{1}{3}x + \frac{\pi}{6}$ ✓

ie. $6y = -2x + \pi$

$2x + 6y - \pi = 0$

iii. $y = \tan x$

$y' = \sec^2 x$

when $x = \frac{\pi}{4}$, $y' = \frac{1}{\cos^2\frac{\pi}{4}} = \frac{1}{(\frac{1}{\sqrt{2}})^2}$

$= 2$

$\therefore$ equ. is $y - 1 = 2(x - \frac{\pi}{4})$ ✓

ie. $y = 2x - \frac{\pi}{2} + 1$

$2y = 4x - \pi + 2$

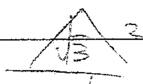
$4x - 2y - \pi + 2 = 0.$

3. $2y + x = 10$

$2y = 10 - x \Rightarrow y = 5 - \frac{x}{2}$

$\therefore m_1 = -\frac{1}{2}$

$\therefore m_2 = -\frac{1}{2}$



$y = \cos x$

$y' = -\sin x$

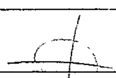
when $y' = -\frac{1}{2} \Rightarrow \frac{1}{2} = \sin x$

$\therefore x = \frac{\pi}{6}$

when $x = \frac{\pi}{6}$, $y = \frac{\sqrt{3}}{2}$

$\therefore (x, y)$ is $(\frac{\pi}{6}, \frac{\sqrt{3}}{2})$ ✓

4. $y = \cos 2x$



$y' = -2\sin 2x$

when $y' = -1$

$2\sin 2x = 1$

$\sin 2x = \frac{1}{2}$ [$0 \le 2x \le 4\pi$] →

$\therefore x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

$\therefore y = \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$

$\therefore$ pnts are $\left(\frac{\pi}{12}, \frac{\sqrt{3}}{2}\right) \left(\frac{5\pi}{12}, -\frac{\sqrt{3}}{2}\right) \cancel{\left(\frac{13\pi}{12}, \frac{\sqrt{3}}{2}\right)}$
$\cancel{\left(\frac{17\pi}{12}, -\frac{\sqrt{3}}{2}\right)}$.

5. $y = \tan x$

$y' = \sec^2 x$

$m_1 = \sec^2 x$

$\therefore m_2 = -\frac{1}{\sec^2 x} = -\cos^2 x$

when $m_2 = \cancel{-\cos^2 x}\ -\frac{1}{2}$

$\cos^2 x = \frac{1}{2}$

$\cos x = \frac{1}{\sqrt{2}}, \cancel{-\frac{1}{\sqrt{2}}}$ (cos in quad ①)

$\therefore x = \frac{\pi}{4}$

when $x = \frac{\pi}{4}$, $y = 1$

$\therefore \left(\frac{\pi}{4}, 1\right)$ is pnt. ✓

6. $y = \sin 2x - \sqrt{3}x$

$y' = 2\cos 2x - \sqrt{3}$

when $y' = 0$, $2\cos 2x = \sqrt{3}$

$\cos 2x = \frac{\sqrt{3}}{2}\ [0 \le 2x \le 4\pi]$

$2x = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$

$x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

$y'' = -4\sin 2x$

$f''\left(\frac{\pi}{12}\right) = -4\sin\frac{\pi}{6} < 0$ $\therefore$ max. tp at $x = \frac{\pi}{12}$ ✗min

$f''\left(\frac{11\pi}{12}\right) = -4\sin\frac{11\pi}{6} > 0$ $\therefore$ min tp. at $x = \frac{11\pi}{12}$ ✗max

$f''\left(\frac{13\pi}{12}\right) = -4\sin\frac{13\pi}{6} < 0$ $\therefore$ max. tp. at $x = \frac{13\pi}{12}$ ✗min

$f''\left(\frac{23\pi}{12}\right) = -4\sin\frac{23\pi}{6} > 0$ $\therefore$ min tp at $x = \frac{23\pi}{12}$ ✗max

7i. $y = x - 2\sin x$

$y' = 1 - 2\cos x$

when $y' = 0$

$2\cos x = 1\ [0 \le x \le 2\pi]$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

$y'' = 2\sin x$

$f''\left(\frac{\pi}{3}\right) = 2\sin\frac{\pi}{3} > 0$

$f''\left(\frac{5\pi}{3}\right) = 2\sin\frac{5\pi}{3} < 0$

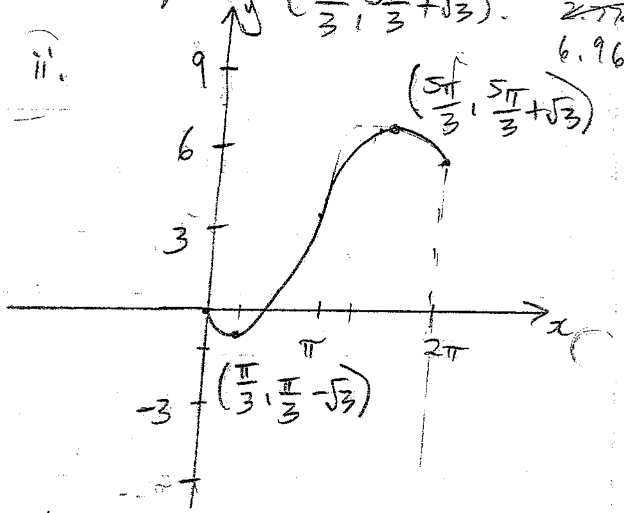
when $x = \frac{\pi}{3}$, $y = \frac{\pi}{3} - 2\sin\frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3}$

" $x = \frac{5\pi}{3}$, $y = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3}$

$\therefore$ min. tp at $\left(\frac{\pi}{3}, \frac{\pi}{3} - \sqrt{3}\right)$ -0.684

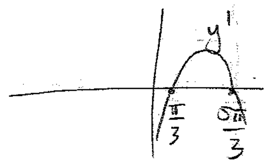
max tp at $\left(\frac{5\pi}{3}, \frac{5\pi}{3} + \sqrt{3}\right)$. 6.968

ii.



when $x = 0$, $y = 0$

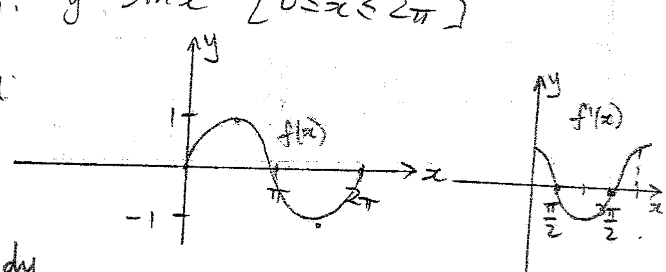
when $x = 2\pi$, $y = 2\pi - 2\sin 2\pi = 2\pi$



iii. $\frac{dy}{dx} \ge 0$ when $\frac{\pi}{3} \le x \le \frac{5\pi}{3}$

$\cancel{\frac{\pi}{3} < x < \frac{5\pi}{3}}$

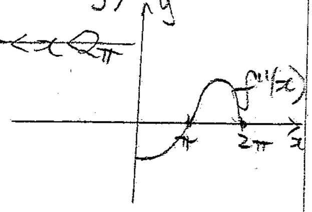
8i. $y = \sin x\ [0 \le x \le 2\pi]$



$\frac{dy}{dx} > 0$ when $0 \le x \le \frac{\pi}{2}$, $\frac{3\pi}{2} < x < 2\pi$ ✓

ii. $\frac{dy}{dx} < 0$ when $\frac{\pi}{2} < x < \frac{3\pi}{2}$ (only)

$\frac{d^2y}{dx^2} > 0$ ~~when $\pi < x < 2\pi$~~



1i. $\int \cos 2x\,dx = \frac{1}{2}\sin 2x + c$ ✓

ii. $\int 4\sin 3x\,dx = -\frac{4}{3}\cos 3x + c$ ✓

iii. $\int 8\sec^2 5x\,dx = \frac{8}{5}\tan 5x + c$ ✓

iv. $\int (6x + \sin \pi x)\,dx = 3x^2 - \frac{1}{\pi}\cos \pi x + c$ ✓

v. $\int \cos \frac{x}{2}\,dx = 2\sin \frac{x}{2} + c$ ✓

vi. $\int (x^2 + \sec^2 \frac{\pi}{2}x)\,dx = \frac{1}{3}x^3 + \frac{2}{\pi}\tan \frac{\pi}{2}x + c$ ✓

vii. $\int \frac{1}{2}\sin 4x\,dx = -\frac{1}{8}\cos 4x + c$ ✓

viii. $\int (2\sec^2 x + \cos 2x)\,dx$
$= 2\tan x + \frac{1}{2}\sin 2x + c.$ ✓

ix. $\int (1 + \sin x)\,dx = x - \cos x + c$ ✓

x. $\int (1 - \sec^2 \frac{1}{2}x)\,dx = x - 2\tan \frac{1}{2}x + c$ ✓

xi. $\int (\pi + \cos x)\,dx = \pi x + \sin x + c$ ✓

xiii. $\int (\pi - \sin \pi x)\,dx = \pi x + \frac{1}{\pi}\cos \pi x + c$ ✓

2i. $\int_0^{\frac{\pi}{4}} \sin x\,dx = \left[-\cos x\right]_0^{\frac{\pi}{4}} = -\cos\frac{\pi}{4} + \cos 0$
$= -\frac{1}{\sqrt{2}} + 1 = -\frac{\sqrt{2}}{2} + 1 = \frac{-\sqrt{2}+2}{2}$ ✓

ii. $\int_0^1 \sec^2 3x\,dx = \left[\frac{1}{3}\tan 3x\right]_0^1$
$= \frac{1}{3}\tan 3 - \frac{1}{3}\tan 0$
$= \frac{1}{3}\tan 3. = -0.0475$

iii. $\int_0^2 4\cos x\,dx = \left[4\sin x\right]_0^2$
$= 4\sin 2 - 4\sin 0 = 4\sin 2. = 3.637$
(radians)

iv. $\int_0^{\frac{\pi}{2}} (1 + \cos x)\,dx = \left[x + \sin x\right]_0^{\frac{\pi}{2}}$
$= \frac{\pi}{2} + \sin\frac{\pi}{2} - 0 - \sin 0 = \frac{\pi}{2} + 1$ ✓

v. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\,dx = \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$
$= -\cos\frac{\pi}{2} - \sin\frac{\pi}{2} + \cos\frac{\pi}{4} + \sin\frac{\pi}{4}$
$= -1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = -1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}$
$= \frac{-2 + 2\sqrt{2}}{2} = -\frac{\cancel{2}(1 - \sqrt{2})}{\cancel{2}}$
$= \sqrt{2} - 1$ ✓

vi. $\int_0^{2\pi} \sin 3x\,dx = \left[-\frac{1}{3}\cos 3x\right]_0^{2\pi}$
$= -\frac{1}{3}\cos 6\pi + \frac{1}{3}\cos 0$
$= -\frac{1}{3} + \frac{1}{3} = 0$ ✓

3. $\frac{dy}{dx} = \sin 2x$
$\therefore y = -\frac{1}{2}\cos 2x + c$
when $x = \pi, y = 3$
$\Rightarrow 3 = -\frac{1}{2}\cos 2\pi + c$
$3 = -\frac{1}{2} + c \Rightarrow \therefore c = 3\frac{1}{2}$
$\therefore$ equ. is $y = -\frac{1}{2}\cos 2x + 3\frac{1}{2}.$ ✓

4i] $y = \sin x$ — ①
$y = \cos x$ — ②
Sub ① in ② $\Rightarrow \sin x = \cos x$
$\tan x = 1$
$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$
Sub in ①, when $x = \frac{\pi}{4}, y = \frac{1}{\sqrt{2}}$
$x = \frac{5\pi}{4}, y = -\frac{1}{\sqrt{2}}$ ✓
$\therefore A$ is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$, B is $\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$.

ii] $A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx$
$= \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$
$= -\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} + \cos\frac{\pi}{4} + \sin\frac{\pi}{4}$
$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}\ u^2$ ✓

5. $V = \pi\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$
$= \pi\left[\tan x\right]_0^{\frac{\pi}{4}} = \pi(1 - 0)$
$= \pi\ u^3.$ ✓