Randwick Boys' High School
Mathematics Department

EXTENSION 1

Year 12

HALF YEARLY EXAMINATION

MAY 2005

Time Allowed : 90 minutes

Examiner: C. Rizos

INSTRUCTIONS :

Candidates may attempt all questions.
All necessary working should be shown in each question.
Full marks may not be awarded for careless or badly set out work.
Hand this question booklet in with your answers.

ANSWER EACH QUESTION ON SEPARATE PAGES

NAME : ______________________

TEACHER : ______________________

QUESTION 1 : Mark

(a) Find the solution (i) $|2x-5|>5$ 2

(ii) $\frac{4}{t}<t$ 2

(b) By writing $0\cdot2\dot{4}$ as an infinite series express it as a fraction. 2

(c) Evaluate $\int_2^5 (x^2+\frac{3}{x^2}-3)dx$ 2

(d) Find an approximation to $\int_0^2 2^x dx$, using Simpson's Rule with five function values. Give your answer in exact form 4

QUESTION 2 :

(a) Simplify : (i) $\frac{2^{n+2}-2^{n-1}}{2^{n+2}+2^{n-1}}$. 2

(ii) $\frac{12^x \times 6^x}{2^{3x}}$ 2

(b) (i) Find the area contained between the curve $y=\sqrt{x}$, the line $2x+y=10$ and the X axis.
(Hint: the curve and the line intersect at $x=4$) 4

(ii) Find the volume generated when the upper segment of the circle $x^2+y^2=6$ from $(-\sqrt{2},2)$ to $(\sqrt{2},2)$ is rotated about the X axis.

Leave your answer in surd form. 4

QUESTION 3 : **Mark**

(a) Find the indefinite integral of (i) $\dfrac{x^3 - 2x}{x^3}$ 2

(ii) $(6x - 5)^3$ 2

(b) A family of 8 sit around a table. Steven, Tuan and Gordon want to sit as a group of three.

(i) How many arrangements with this restriction are there? 2

(ii) What is the probability of this arrangement occurring? 2

(c) There are two bags each with white and black balls in them:

Bag A contains n+1 white balls and n-1 black balls.

Bag B contains n-1 white balls and n+1 black balls.

Each time a white ball is drawn from either bag it is put back into bag A. Black balls, however, are put back into the bag they came from. The first ball is drawn from bag A, and the second from bag B.

(i) Show that the probability that both balls are white is

$$\frac{(n+1)(n-1)}{4n^2}$$ 2

(ii) Two more balls are then drawn (from bag A, then bag B, as above). Show that the probability that all four balls are white is

$$\frac{(n+1)(n+2)(n-1)(n-2)}{4n^2(2n+1)(2n-1)}$$ 2

QUESTION 4 :

(a) Consider the function $y = \frac{4}{x-1} + x$

(i) State the domain of the function. 1

(ii) Describe the tendencies of the curve as x approaches ∞ and as x approaches $-\infty$. 2

(iii) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and show that the function has no points of inflexion. 3

(iv) Sketch the function showing clearly where any maximum and/or minimum turning points occur. 2

(b) For a given series $T_{n+1} - T_n = 7, T_1 = 3$, find the value of S_{100}, where $S_n = T_1 + T_2 + T_3 + \dots\dots + T_n$. 2

(c) Show that the equation of the tangent at $T(-2t, t^2)$ on the parabola $y = \frac{1}{4}x^2$ is given by $y + tx + t^2 = 0$. 2

QUESTION 5 :

(a) Alex decided that on her birthday each year she will deposit an amount **$M** into an investment account which pays $r\%$ interest, compounded annually.

Let $\$A_n$ be the amount that her investment is worth at the end of n years, after interest has been added, but before she makes her deposit for the next year.

(i) Show that $A_n = \dfrac{MR(R^n - 1)}{R - 1}$ where $R = 1 + \dfrac{r}{100}$.

Alex decided to deposit $1 000 each year and she chose an account which paid **8%** per annum interest, compounded annually. She made her first deposit on her 21st birthday in 1980.

(ii) Show that on her 31st birthday in 1990 her investment was worth $15 645 to the nearest dollar.

As part of her preparation for her eventual retirement Alex thought that she should try to accumulate $250 000. So, on her birthday in 1990 she transferred her funds to a new investment account which paid **10%** per annum interest which compounded annually. She also decided to increase her deposits to $2 000 each year. She made her first deposit into the new account on the same day and she will continue each year until she reaches her goal.

(iii) On which birthday will Alex accumulate at least $250 000? 6

51st

(b) Let each different arrangement of the letters of ESTEEM be called a word.

(i) How many words are possible if all letters are used?

(ii) If one of these words is chosen at random what is the probability that three Es are together? 2

(c) Use the principle of mathematical induction to prove that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$ 4

END OF PAPER

YR 12 EXTENSION HALF YEARLY 2005

Q1 (a)(i) $|2x-5| > 5$

$2x-5 < -5$ ✓ OR $2x-5 > 5$ ✓

$x < 0$ OR $x > 5$

(ii) $\frac{4}{t} < t$

$4t < t^3$ ✓

$t(t^2-4) > 0$

$-$ (−2) $+$ (0) $-$ (2) $+$

OR $\frac{4}{t} < t$

$\frac{4}{t} - t < 0$ ✓

$\frac{4-t^2}{t} < 0$

$\frac{(2-t)(2+t)}{t} < 0$

$\therefore -2 < t < 0$ and $t > 2$ ✓

(b)

$0.2\dot{4} = 0.2444\ldots$

$= 0.2 + 0.0\dot{4}$ ✓

$0.0\dot{4} \times 10 = 0.\dot{4}$

$10 \times 0.0\dot{4} - 0.0\dot{4} = 0.4$

$= 9 \times 0.0\dot{4}$

$\therefore 0.0\dot{4} = \frac{0.4}{9}$

$= \frac{4}{90}$

$\therefore 0.2\dot{4} = \frac{2}{10} + \frac{4}{90}$ ✓

$= \frac{11}{45}$

(c) $\int_2^5 \left(x^2 + \frac{3}{x^2} - 3\right) dx = \left(\frac{x^3}{3} - \frac{3}{x} - 3x\right)_2^5$ ✓

$= 30.9$ ✓

(d) $y = 2^x$

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	1	$\sqrt{2}$	2	$2\sqrt{2}$	4

✓ ✓

$\int_0^2 2^x dx \doteq \frac{1-0}{6}\left[1 + 4\sqrt{2} + 2 + 2 + 8\sqrt{2} + 4\right]$ ✓

$= \frac{1}{6}(9 + 12\sqrt{2})$

$A \doteq \frac{1}{2}(3 + 4\sqrt{2})$ units^2 ✓

OR

$\int_0^2 2^x dx = \frac{\frac{1}{2}}{3}(1 + (4\sqrt{2}) + 2) + \frac{\frac{1}{2}}{3}(2 + (8\sqrt{2}) + 4)$ ✓

$= \frac{1}{6}(9 + 12\sqrt{2})$

$A \doteq \frac{1}{2}(3 + 4\sqrt{2})u^2$ ✓

Q2. (a)(i) $\frac{2^{n+2} - 2^{n-1}}{2^{n+2} + 2^{n-1}} = \frac{2^3 \cdot 2^{n-1} - 2^{n-1}}{2^3 \cdot 2^{n-1} + 2^{n-1}}$ ✓

$= \frac{2^{n-1}(2^3-1)}{2^{n-1}(2^3+1)}$ ✓

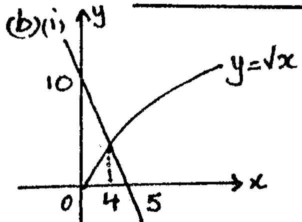
$= \frac{7}{9}$

(ii) $\frac{12^x \times 6^x}{2^{3x}} = \frac{(2^2 \times 3)^x \times 2^x \times 3^x}{2^{3x}}$ ✓

$= \frac{2^{2x} \times 3^x \times 2^x \times 3^x}{2^{3x}}$

$= 3^{2x}$ ✓ (or 9^x)

(b)(i)



✓diagram

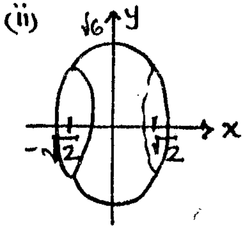
$A = \left|\int_0^4 \sqrt{x}\,dx\right| + \frac{1}{2} \times 1 \times 2$ ✓ ✓

$= \left|\left(\frac{2}{3}x^{3/2}\right)_0^4\right| + 1$

$= \left|\frac{16}{3} + 1\right|$

$A = 6\frac{1}{3}$ units^2 ✓

(ii)



$V = \pi \left|\int_{-\sqrt{2}}^{\sqrt{2}} y^2 dx\right|$

$= \pi \left|\int_{-\sqrt{2}}^{\sqrt{2}} (6 - x^2)\right| dx$ ✓ ✓

$= \left|\pi\left(6x - \frac{x^3}{3}\right)_{-\sqrt{2}}^{\sqrt{2}}\right|$ ✓

$= \pi\left|\left(6\sqrt{2} - \frac{2}{3}\sqrt{2} + 6\sqrt{2} - \frac{2}{3}\sqrt{2}\right)\right|$

$= \frac{32\sqrt{2}}{3}\pi$ units^3 ✓

Q3 (a) (i) $\int \frac{x^3-2x}{x^3}\,dx = \int 1-2x^{-2}\,dx$ ✓

$= x + \frac{2}{x} + C$ ✓

(ii) $\int (6x-5)^3\,dx = \frac{(6x-5)^4}{4\times 6} + C$ ✓

$= \frac{(6x-5)^4}{24} + C$ ✓

(b) 8 Student 3 together

(i) 3! ways of arranging 3 students ✓

Now have 6 in group ⇒ 5! ways of ✓ arranging in a circle

⇒ 3!5! = 720 arrangements

(ii) 7! arrangements without restrictions ✓

$\therefore P(\text{Occurs}) = \frac{720}{7!}$

$= \frac{1}{7}$ ✓

(c)

Bag A	
n+1	W
n-1	B

Bag B	
n-1	W
n+1	B

(i) $P(W_A) = \frac{n+1}{(n+1)+(n-1)}$

$= \frac{n+1}{2n}$ ✓

$P(W_B) = \frac{n-1}{2n}$ ✓

$\therefore P(W_A W_B) = \frac{(n+1)(n-1)}{4n^2}$

(ii) $P(W_A W_B) = \frac{(n+1)(n-1)}{4n^2}$

White ball from Bag B put in Bag A ⇒ (n+2) W in A, (2n+1) balls in A ✓

(n-2) W in B, (2n-1) balls in B ✓

$\therefore P(W_A W_B W_A W_B) = \frac{(n+1)(n-1)(n+2)(n-2)}{4n^2(2n+1)(2n-1)}$

Q4 (a) $y = \frac{4}{x-1} + x$

(i) $x \in \mathbb{R}$, $x \neq 1$ ✓

(ii) $x \to \infty$ $y \to x$ ✓

$x \to -\infty$ $y \to -x$ ✓

(iii) $y' = \frac{-4}{(x-1)^2} + 1$ ✓

$y'' = \frac{8}{(x-1)^3}$ ✓

$y'' \neq 0$ for any x ✓

(iv) S.P occur $y' = 0$

$\therefore \frac{4}{(x-1)^2} = 1$

$(x-1)^2 = 4$

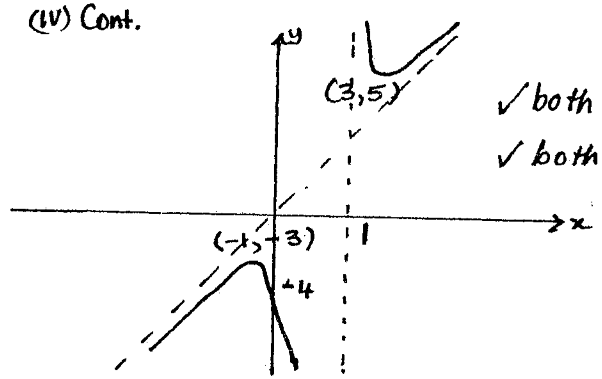
$x - 1 = -2, 2$

$x = -1, 3$

$\therefore$ SP $(-1,-3)$ $y'' < 0$ ∴ Rel Max

SP $(3,5)$ $y'' > 0$ ∴ Rel Min

(iv) Cont.



✓ both asymptotes

✓ both parts of curve.

(b) $T_{n+1} - T_n = 7$ $T_1 = 3$

$S_n = T_1 + T_2 + T_3 + \cdots + T_n$

An AP ✓ $S_n = \frac{n}{2}(2a+(n-1)d)$

3, 10, 13 .....

$S_{100} = \frac{100}{2}(2\times 3 + 99\times 7)$ ✓

$= 34950$

(c) Gradient of tangent is y'

$y = \frac{1}{4}x^2$

$y' = \frac{1}{2}x$

$x = -2t$ $m_T = -t$ ✓

$(-2t, t^2)$ $m = -t$ ⇒

Tangent $y - t^2 = -t(x+2t)$ ✓

$y - t^2 = -tx - 2t^2$

$y + tx + t^2 = 0$

Q5 (a) (i) $A_1 = M \times R \qquad R = 1 + \frac{r}{100}$

$A_2 = A_1 \times R + M \times R$

$= M \times R^2 + M \times R$

$A_3 = M \times R^3 + M \times R^2 + M \times R$ ✓

$\vdots \qquad \vdots$

$A_n = M \underbrace{(R + R^2 + R^3 + \ldots + R^n)}_{GP}$

$a = R \quad r = R$

$S_n = \frac{R(R^n - 1)}{R-1}$ ✓

$\therefore A_n = \frac{MR(R^n - 1)}{R-1}$

(ii) $M = \$1000 \quad R = 1.08 \quad n = 10$ ✓

$A_{10} = \frac{1000 \times 1.08(1.08^{10} - 1)}{1.08 - 1}$

$A_{10} = \$\,15\,645$ ✓

(iii) $A_1 = (15645 + 2000) \times 1.1$

$A_2 = 17645 \times 1.1^2 + 2000 \times 1.1$

$A_3 = 17645 \times 1.1^3 + 2000 \times 1.1^2 + 2000 \times 1.1$

$\vdots \qquad \vdots \qquad \vdots$

$A_n = 17645 \times 1.1^n + 2000\,\overset{GP}{(1.1 + 1.1^2 + 1.1^3 + \ldots + 1.1^n)}$ ✓

$250\,000 = 17645 \times 1.1^n + 2000 \left(\frac{1.1(1.1^n - 1)}{1.1 - 1}\right)$

$= 17645 \times 1.1^n + 22000\,(1.1^n - 1)$

$= 17645 \times 1.1^n + 22000 \times 1.1^n - 22000$

$272000 = 1.1^n(17645 + 22000)$

$1.1^n = 6.86\ldots \Longrightarrow 1.1^{20} = 6.727\ldots$ $\therefore$ After 21 years, ie on 51st birthday. ✓

(b) ESTEEM

(i) 6 letters 3 Es

$\therefore \frac{6!}{3!}$ words $= 120$ ✓

(ii) Es together $\therefore$ 4 choices

EEE S T M

$4!$ ways $= 24$

$\therefore P(3 \text{ Es together}) = \frac{24}{120}$ ✓

$= \frac{1}{5}$

(c) $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$

- Test for $n=1$ $\quad LHS = \sum_{r=1}^{1} \frac{1}{r(r+1)(r+2)}$

$= \frac{1}{6}$

$RHS = \frac{1}{4} - \frac{1}{2(1+1)(1+2)}$

$= \frac{1}{6}$

$= LHS$

$\therefore$ True for $n=1$ ✓

- Assume true for $n=k$:

$\sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$

- To prove true for $n=k+1$

$\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \frac{1}{(k+1)(k+2)(k+3)} + \sum_{r=1}^{k} \cdots$ ✓

$= \frac{1}{(k+1)(k+2)(k+3)} + \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$

$= \frac{1}{4} - \left[\frac{(k+3) - 2}{2(k+1)(k+2)(k+3)}\right]$ ✓

$= \frac{1}{4} - \frac{k+1}{2(k+1)(k+2)(k+3)}$

$= \frac{1}{4} - \frac{1}{2[(k+1)+1][(k+1)+2]}$

$\therefore$ True for $n=k+1$, when assume true for $n=k$

Since shown true for $n=1$ and $n=k+1$ when assumed true for $n=k$ then it must hold true for $n=2$ $\therefore n=3$ and so on. ✓