ROSE BAY SECONDARY COLLEGE

YEAR 12 MATHEMATICS
ASSESSMENT TASK 3
HALF YEARLY EXAM

April 26th 2007

Time Allowed : 2 hours plus 5 minutes reading time

Instructions:
- All questions are of equal value
- Begin each question on a NEW PAGE
- Read each question carefully and ensure that all necessary working is shown
- Diagrams need to be a reasonable size and a ruler should be used
- Marks may be deducted for poor presentation, omission of clear reasoning and incomplete arguments

TOTAL MARKS: 84

Question 1 (Begin a new page) — Marks

a) Find the value of $\log_3 81$ — 1

b) Find the domain and range of $y = e^{2x} - 4$ — 2

c) Evaluate $\lim\limits_{x \to 3} \dfrac{x^2 - 6x + 9}{x - 3}$ — 2

d) Solve for x: $9^x = \dfrac{1}{27}$ — 2

e) Find the primitive function of $\dfrac{1}{(3x-2)^4}$ — 2

f) For what value of m does $x^2 + (m+3)x + 4m = 0$ have real roots? — 3

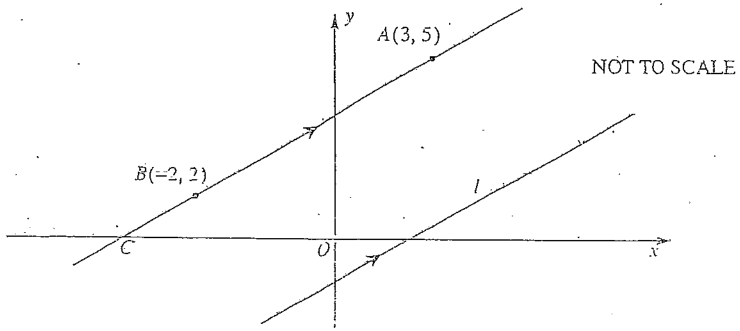
Question 2 (Begin a new page)

a) Given that $\log_a b = 3\cdot5$ and $\log_a c = 0\cdot35$, find the value of

i) $\log_a\left(\dfrac{c}{b}\right)$ — 1

ii) $\log_a (bc)^2$ — 2

b)



The diagram shows the points A(3,5) and B(-2,2). The line AB cuts the x axis at the point C. The line l has equation $3x - 5y - 8 = 0$ and is parallel to AB.

i) Find the gradient of the line AB — 1

ii) Show that the equation of AB is $3x - 5y + 16 = 0$. — 1

iii) Find the coordinates of C — 1

iv) Write down the size of $\angle ACO$ correct to the nearest degree. — 1

v) Find the length of the interval AB. — 1

vi) Find the perpendicular distance of B from the line l. — 2

vii) Find the area of the triangle formed by the points A and B and any point P, on the line l. — 2

Question 3 (Begin a new page)

a) i) Write the equation of the parabola $y^2 - 6y + 25 = 8x$ in the form $(y-k)^2 = 4a(x-h)$ 1

ii) Hence find the focal length and the coordinates of the focus 2

b) For a certain geometric sequence, the third and sixth terms are -24 and 3 respectively. Find the sum of the first eight terms of this sequence. 3

c) Differentiate the following

i) $\sqrt{5x^2-4}$ 2

ii) $\dfrac{\ln x}{x}$ 2

iii) e^{2x^3-6x+1} 2

Question 4 (Begin a new page)

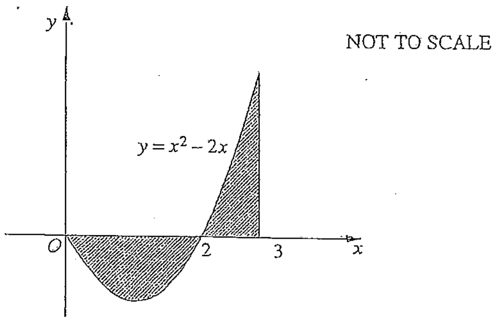
a) Find

i) $\int e^{3x}\,dx$ 1

ii) $\int \dfrac{dx}{5x-6}$ 1

b) Evaluate $\int_0^2 \dfrac{x}{x^2+4}\,dx$ 3

c)

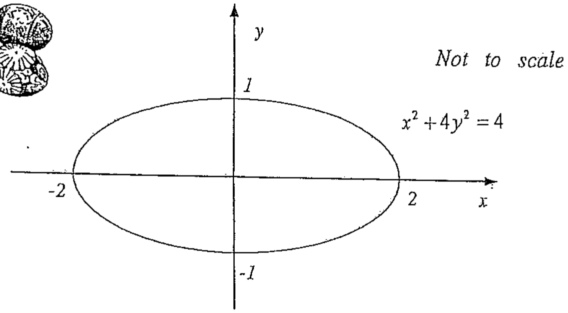


The diagram shows the graph of $y = x^2 - 2x$ for $0 \le x \le 3$.
Find the total area of the shaded section. 3

d) Find the equation of the normal to the curve $y = e^{\frac{x}{2}}$ at the point $(2, e)$.
Write the equation in general form. 4

Question 5 (Begin a new page)

a) A solid chocolate Easter egg can be generated by rotating the area bounded by the ellipse $x^2 + 4y^2 = 4$ around the x axis



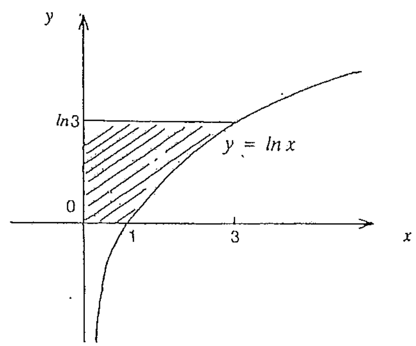
Calculate the volume of chocolate required to make this solid Easter egg. 4

b) The curve $y = f(x)$ has gradient function $f'(x) = kx - 2$.
The curve has gradient 10 at the point (2,9).

i) Findthe value of k 1

ii) Find the equation of the curve $y = f(x)$. 2

c)



The diagram shows the area bounded by the graph $y = \ln x$, the coordinate axes and the line $y = \ln 3$.

i) Find the shaded area 3

ii) Hence find the exact value of $\int_1^3 \ln x\,dx$ 2

Question 6 (Begin a new page)

a)

x	1	2	3	4	5
$x\ln x$	0	1.386	3.296	5.545	8.047

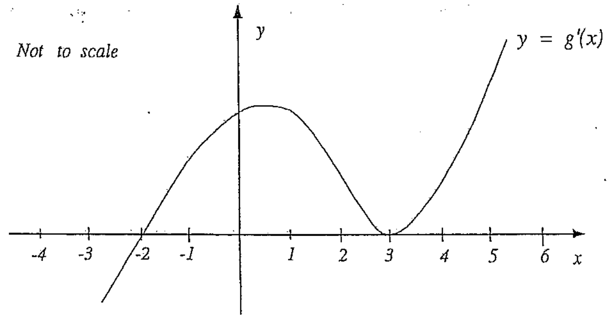
The table shows the values of $x\ln x$ for 5 values of x
Find the approximate value for $\int_1^5 x\ln x \, dx$ using Simpsons rule with the 5 function values in the table. Express your answer correct to 2 decimal places. 3

b) For the curve $y = 2x^3 - 6x^2 - 18x + 1$

i) Find the stationary points and determine their nature. 3
ii) Find any points of inflexion 2
iii) For what values of x is the curve increasing? 1
iv) Sketch the curve in the domain $-2 \le x \le 5$ 3

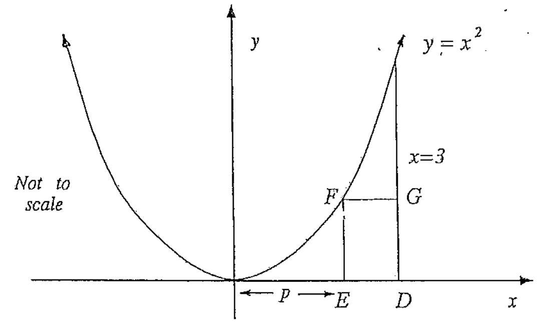
Question 7 (Begin a new page)

a) The graph shows the gradient function of a curve $y = g(x)$



i) Write down the x coordinate of the turning point on $y = g(x)$ and state whether it is a (local) maximum or minimum 1
ii) At $x = 3$ on the curve there is a horizontal point of inflexion. Justify this statement by reference to the graph. 1
iii) For what values of x is $y = g(x)$ increasing? 1
iv) Sketch a possible graph of $y = g(x)$. 2

b) A rectangle DEFG is drawn under the curve $y = x^2$ with one side along the line $x = 3$ as shown in the diagram below.
The side EF is p units from the origin.



i) Show that the area of the rectangle, A square units, is given by the expression

$$A = 3p^2 - p^3$$ 2

ii) Find the largest possible area of this rectangle 5

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QUESTION 1

a) $\log_3 81 = 4$ (since $3^4 = 81$)

b) Domain: all real values of x

Range: $y > -4$

c) $\lim_{x \to 3} \frac{x^2-6x+9}{x-3}$

$= \lim_{x\to 3} \frac{(x-3)^2}{(x-3)} = 3-3$

$= 0$

d) $9^x = \frac{1}{27}$

$\therefore 3^{2x} = 3^{-3}$

$\therefore 2x = -3$

$x = -\frac{3}{2}$

e) $\frac{1}{(3x-2)^4} = (3x-2)^{-4}$

$\therefore$ Primitive $= \frac{(3x-2)^{-3}}{3\times -3} + c$

$= -\frac{1}{9}(3x-2)^{-3} + c$

or $= \frac{-1}{9(3x-2)^3} + c$

f) $x^2 + (m+3)x + 4m = 0$

$\Delta = b^2 - 4ac$

$= (m+3)^2 - 4.1.4m$

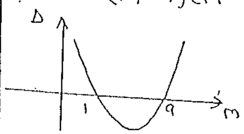
$= m^2 + 6m + 9 - 16m$

$= m^2 - 10m + 9$

Now Real Roots exist when $\Delta \geq 0$

ie $m^2 - 10m + 9 \geq 0$

$(m-9)(m-1) \geq 0$



$\therefore m \leq 1, m \geq 9$

QUESTION 2.

a) i) $\log_a\left(\frac{c}{b}\right) = \log_a c - \log_a b$

$= 0.35 - 3.5$

$= -3.15$

ii) $\log_a (bc)^2 = 2\log_a(bc)$

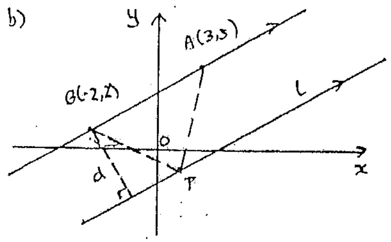
$= 2[\log_a b + \log_a c]$

$= 2[3.5 + 0.35]$

$= 2 \times 3.85$

$= 7.7$

b)



i) $\text{Grad}_{AB} = m_1 = \frac{y_2-y_1}{x_2-x_1}$

$= \frac{5-2}{3--2}$

$= \frac{3}{5}$

ii) For AB $(x_1,y_1) = (3,5)$

$m_1 = \frac{3}{5}$

$\therefore$ Eq^n of AB

$(y-y_1) = m(x-x_1)$

$y-5 = \frac{3}{5}(x-3)$

$5y - 25 = 3x - 9$

$\therefore 3x - 5y + 16 = 0$ as req'd.

iii) x int occurs where $y = 0$

$\therefore 3x - 5\times 0 + 16 = 0$

$\therefore 3x + 16 = 0$

$3x = -16$

$x = -5\frac{1}{3}$

$\therefore C\left(-5\frac{1}{3}, 0\right)$

iv) Let $\angle ACO = \theta$

$\therefore \tan\theta = m_1$

$\tan\theta = \frac{3}{5}$

$\theta = 30°57'49.52''$

$= 31°$ (nr deg)

v) $AB^2 = (y_2-y_1)^2 + (x_2-x_1)^2$

$= (3--2)^2 + (5-2)^2$

$= 5^2 + 3^2$

$= 25 + 9$

$= 34$

$\therefore AB = \sqrt{34}$ $(AB > 0)$

vi) Let perp dist be d

$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

L is $3x - 5y - 8 = 0$

$\therefore a = 3 \quad b = -5 \quad c = -8 \quad x_1 = -2 \quad y_1 = 2$

$\therefore d = \frac{|3\times -2 + -5\times 2 + -8|}{\sqrt{3^2 + (-5)^2}}$

$= \frac{|-6-10-8|}{\sqrt{34}}$

$= \frac{|-24|}{\sqrt{34}}$

$= \frac{24}{\sqrt{34}} = \frac{24\sqrt{34}}{34} = \frac{12\sqrt{34}}{17}$

vii) $A = \frac{1}{2}bh$

$= \frac{1}{2} \times \sqrt{34} \times \frac{12\sqrt{34}}{17}$

$= \frac{6\times 34}{17}$

$= 12\, u^2$

QUESTION 3

a) i) $y^2 - 6y + 25 = 8x$

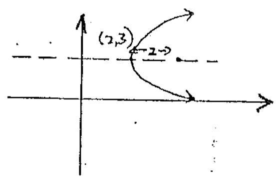
$\therefore y^2 - 6y + 9 = 8x - 25 + 9$

$(y-3)^2 = 8x - 16$

$(y-3)^2 = 8(x-2)$

ii) $4a = 8$

$\therefore a = 2$ (Focal length)



Focus is $(4,3)$

b) $T_3 = ar^2 = -24$ ①

$T_6 = ar^5 = 3$ ②

$\frac{②}{①}$

$\therefore \frac{ar^5}{ar^2} = \frac{3}{-24}$

$\therefore r^3 = -\frac{1}{8}$

$r = -\frac{1}{2}$

$\therefore a\left(-\frac{1}{2}\right)^2 = -24$

$a \times \frac{1}{4} = -24$

$a = -96$

$S_n = \frac{a(1-r^n)}{1-r}$

$S_8 = \frac{-96\left(1-\left(-\frac{1}{2}\right)^8\right)}{1-\left(-\frac{1}{2}\right)} \left(= -96 \times \frac{2}{3} \times \frac{255}{256}\right)$

$= -63\frac{3}{4}$

c) i) $\dfrac{d}{dx}\sqrt{5x^2-4}$

$= \dfrac{d}{dx}(5x^2-4)^{\frac{1}{2}}$

$= \frac{1}{2}(5x^2-4)^{-\frac{1}{2}} \times 10x$

$= 5x\,(5x^2-4)^{-\frac{1}{2}}$

$= \dfrac{5x}{\sqrt{5x^2-4}}$

ii) $\dfrac{d}{dx}\left(\dfrac{\ln x}{x}\right)_v^u$ $\quad u = \ln x \quad v = x$, $\quad u' = \frac{1}{x} \quad v' = 1$

$= \dfrac{vu' - uv'}{v^2}$

$= \dfrac{x \times \frac{1}{x} - \ln x \times 1}{x^2}$

$= \dfrac{1 - \ln x}{x^2}$

iii) $\dfrac{d}{dx}\left(e^{2x^3-6x+1}\right)$

$= (6x^2-6)\,e^{2x^3-6x+1}$

$= 6(x^2-1)\,e^{2x^3-6x+1}$

Question 4

a) i) $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + c$

ii) $\int \dfrac{dx}{5x-6} = \frac{1}{5}\ln(5x-6) + c$

b) $\int_0^2 \dfrac{x}{x^2+4}\,dx$

$= \frac{1}{2}\int_0^2 \dfrac{2x}{x^2+4}\,dx$

$= \frac{1}{2}\left[\ln(x^2+4)\right]_0^2$

$= \frac{1}{2}\left[\ln(2^2+4) - \ln(0^2+4)\right]$

$= \frac{1}{2}\left[\ln 8 - \ln 4\right]$

$= \frac{1}{2}\left[\ln \frac{8}{4}\right]$

$= \frac{1}{2}\ln 2$

c) $A = \int_2^3 x^2-2x\,dx + \left|\int_0^2 x^2-2x\,dx\right|$

$= \left[\frac{x^3}{3} - x^2\right]_2^3 + \left|\left[\frac{x^3}{3} - x^2\right]_0^2\right|$

$= \left[(9-9) - (\frac{8}{3}-4)\right] + \left|(\frac{8}{3}-4) - 0\right|$

$= \frac{4}{3} + \left|-\frac{4}{3}\right|$

$= \frac{4}{3} + \frac{4}{3}$

$= \frac{8}{3} = 2\frac{2}{3}\ u^2$

d) $y = e^{x/2}$

$y' = \frac{1}{2}e^{x/2}$

grad tangent at $(2,e)$ $\quad m_1 = \frac{1}{2}e^{2/2} = \frac{1}{2}e = \frac{e}{2}$

$\therefore$ grad normal $m_2 = -\frac{2}{e}$ (since $m_1 \cdot m_2 = -1$)

Equation normal at $(2,e)$

$y - y_1 = m(x - x_1)$

$y - e = -\frac{2}{e}(x-2)$

$ey - e^2 = -2x + 4$

$\therefore 2x + ey - e^2 - 4 = 0$

Question 5

a) $V = \pi\int_a^b y^2\,dx$

Eq^n of ellipse

$x^2 + 4y^2 = 4$

$\therefore 4y^2 = 4 - x^2$

$y^2 = 1 - \frac{1}{4}x^2$

$\therefore V = \pi\int_{-2}^{2} 1 - \frac{1}{4}x^2\,dx$

$= 2\pi\int_0^2 1 - \frac{1}{4}x^2\,dx$

$= 2\pi\left[x - \frac{x^3}{12}\right]_0^2$

$= 2\pi\left[(2 - \frac{8}{12}) - (0)\right]$

$= 2\pi \times \frac{4}{3} = \frac{8\pi}{3}\ u^3$

b) i) $f'(x) = kx - 2$

grad $= 10$ when $x = 2$

$\therefore 10 = k \times 2 - 2$

$2k = 12$

$k = 6$

ii) $\therefore f'(x) = 6x - 2$

$\therefore f(x) = \frac{6x^2}{2} - 2x + c$

$= 3x^2 - 2x + c$ ne

$(2,9)$ lies on curve

$\therefore 9 = 3 \times 2^2 - 2 \times 2 + c$

$9 = 12 - 4 + c$

$1 = c$

$\therefore f(x) = 3x^2 - 2x + 1$

c) i) $A = \int_a^b x\,dy$

Now $y = \log_e x$

$\therefore x = e^y$

$\therefore A = \int_0^{\ln 3} e^y\,dy$

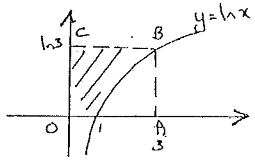
$= \left[e^y\right]_0^{\ln 3}$

$= e^{\ln 3} - e^0$

$= 3 - 1$

$= 2\ u^2$

ii) $\int_1^3 \ln x\,dx$ is area under curve (between curve + x axis)



$\therefore \int_1^3 \ln x$ = Area Rectangle OABC − Area between curve + y axis

$= (3 \times \ln 3) - 2$

$= (3\ln 3 - 2)$

QUESTION 6

a) i) $\int_1^5 x\ln x\,dx \doteq \frac{h}{3}\left[y_0+y_4+2y_2+4(y_1+y_3)\right]$

$\doteq \frac{1}{3}\left[0+8.047+2\times3.296+4(1.386+5.545)\right]$

$\doteq 14.121$

$\doteq 14.12$

b) i) $y = 2x^3-6x^2-18x+1$

$y' = 6x^2-12x-18$

$y'' = 12x-12$

Stat points occur where

$y' = 0$

$6x^2-12x-18 = 0$

$x^2-2x-3 = 0$

$(x-3)(x+1) = 0$

$\therefore x = -1, 3$

$y = 11, -53$

$\therefore$ Stat points occur at

$(-1,11)$ and $(3,-53)$

at $x=-1$. $y'' = 12\times-1-12 < 0$ $\frown$

$\therefore (-1,11)$ is a local Max

at $x=3$ $y'' = 12\times3-12 > 0$ $\smile$.

$\therefore (3,-53)$ is a local Min

ii) Points of inflexion may occur where $y''=0$

$12x-12 = 0$

$12x = 12$

$x = 1$ $y = -21$

check y'' either side

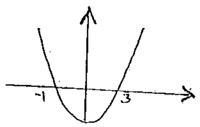
x	0	1	2
y''	−	0	+

y'' changes sign either side of $(1,-21)$ $\therefore$ it is a point of inflexion

iii) Curve is increasing where $y' > 0$

ie $6x^2-12x-18 > 0$

$(x-3)(x+1) > 0$

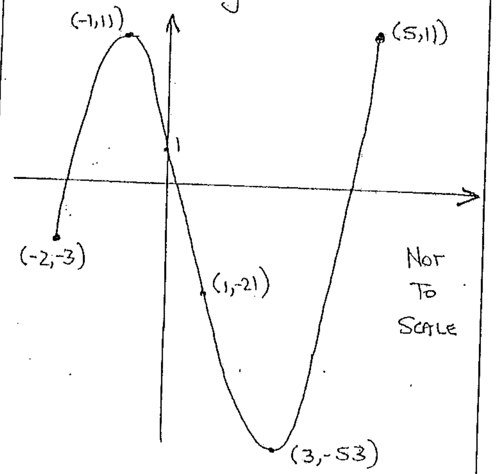


$x > 3$

$x < -1$

iv) at $x=-2$ $y=-3$

at $x=5$ $y=11$



v) Minimum value of the function is -53.

QUESTION 7

a) i) Turning point is at $x=-2$. Gradient changes from negative to positive $\therefore$ It is a local MINIMUM

ii) A horizontal point of inflexion occurs at $x=3$ because the gradient is positive either side of the point (and $y'=0$ at the point)

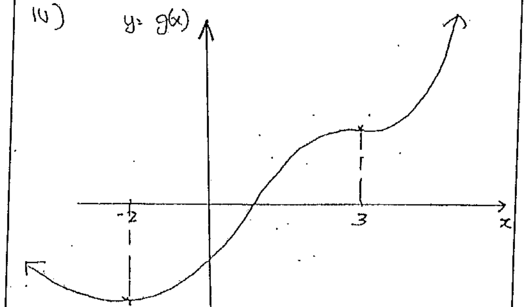
iii) Curve is increasing where $y' > 0$

ie

for $-2 < x < 3$

and $x > 3$

iv)



b) i) $DE = 3-p$

$EF = p^2$

$\therefore$ Area of rectangle $= L\times b$

$= p^2\times(3-p)$

$= 3p^2-p^3$ as req

ii) Max area occurs when

$A' = 0$

ie $\therefore A = 3p^2-p^3$

$A' = 6p-3p^2$

$A'' = 6-6p$

$\therefore 6p-3p^2 = 0$

$3p(2-p) = 0$

$\therefore p=0$ $\quad 2-p=0$

$p=2$

If $p=0$ Area of rect $=0$

$\therefore$ at $p=2$

$A'' = 6-6\times2 < 0$

Local Max occurs when $p=2$

When $p=2$

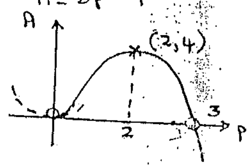
Area $= 3\times2^2-2^3$

$= 12-8$

$= 4u^2$

But you must check that this is actually the largest value for A that can occur in the natural domain (by sketching the graph. nb $x>0$, $A>0$)

$A = 3p^2-p^3$



For $0<p<3$ Max value for A is at $p=2$ ie $A=4$.