NAME :

Centre of Excellence in Mathematics
S201 / 414 GARDENERS RD. ROSEBERY 2018
www.cemtuition.com.au

MOBILE 0412 880 475

PHONE 96663331

YEAR 12 – EXT.2 MATHS

REVIEW TOPIC (SP1)

VOLUMES BY VARIOUS METHODS

(1) By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region $\{(x, y) : 0 \le x \le 1, 0 \le y \le \tan^{-1} x\}$ about the y-axis.

$\boxed{\frac{\pi}{2}(\pi - 2)}$

(2) By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region $\{(x,y): 0 \le x \le 1, e^x \le y \le e\}$ about the line $x = 1$.

$\boxed{\pi(4-e)}$

(3) By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region enclosed by the curve

$\sqrt{x}+\sqrt{y}=\sqrt{a}$

and the coordinate axes about the y-axis.

$$\boxed{\frac{\pi a^3}{15}}$$

(4) By taking slices perpendicular to the axis of rotation, use the method of slicing to find the volume of the solid obtained by rotating the region $\{(x,y): x+y \geq 1, x^2+y^2 \leq 1\}$ about the line $x=1$.

$$\pi\left(\frac{\pi}{2}-\frac{4}{3}\right)$$

(5) By taking strips parallel to the axis of rotation, use the method of cylindrical shells to find the volume of the solid obtained by rotating the region :

$\{(x, y) : 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \cos x\}$ about the line $x = \frac{\pi}{2}$.

$\boxed{2\pi}$

(6) By taking strips parallel to the axis of rotation, use the method of cylindrical shells to find the volume of the solid obtained by rotating the region
$\{(x, y): 0 \leq x \leq 1;\ e^x \leq y \leq e\}$
about the line $y = e$.

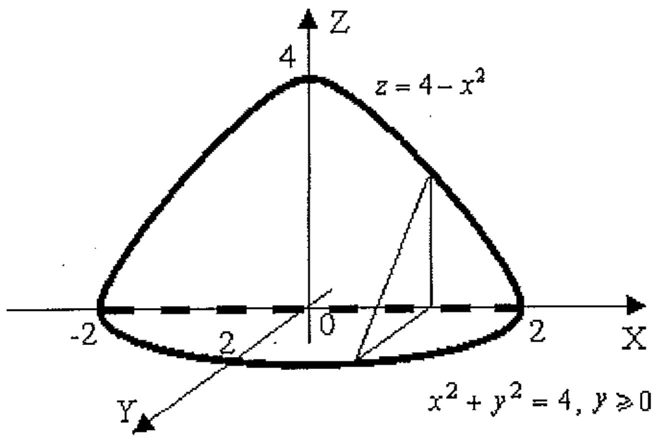
$\frac{\pi}{2}\left(-e^2 + 4e - 1\right)$

(7) The base of a particular solid is the region bounded by the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ between its vertex (2, 0) and the corresponding latus rectum. Every cross-section perpendicular to the major axis is a semicircle with diameter in the base of the solid. Find the volume of the solid.

Hint: The latus rectum of the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ is the line $x = 4$.

$\boxed{16\pi}$

(8) The solid shown has a semicircular base of radius 2 units. Vertical cross-sections perpendicular to the diameter are right-angled triangles whose height is bounded by the parabola $z = 4 - x^2$.



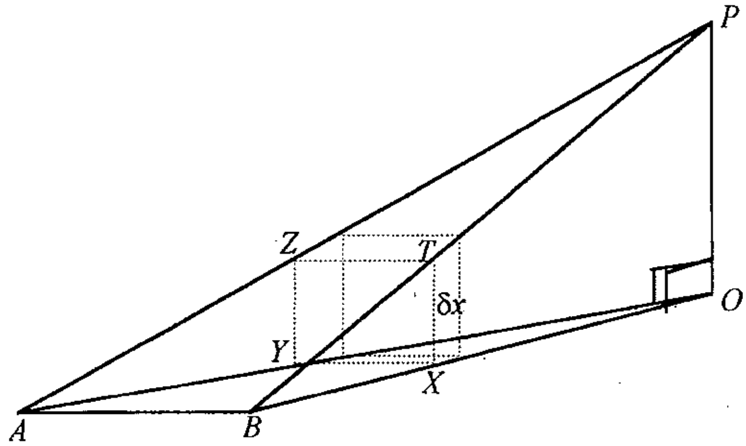
By slicing at right angles to the x- axis, show that the volume of the solid is given by $V = \int_0^2 \left(4 - x^2\right)^{3/2} dx$, and hence calculate this volume.

$\boxed{3\pi}$

(9)

HSC '89

(5) (b)



Let ABO be an isosceles triangle, $AO = BO = r$, $AB = b$.

Let $PABO$ be a triangular pyramid with height $OP = h$ and OP perpendicular to the plane of ABO as in the above diagram.

Consider the slice S of the pyramid of which δx as in the diagram.

The slice S is perpendicular to the plane of ABO at XY with $AB \| XY$ and $XB = a$.

Note that $XT \| OP$.

(i) Show that the volume of S is $\left(\frac{r-a}{r}\right) b \left(\frac{ah}{r}\right) \delta a$ when δa is small. (You may assume that the slice approximately a rectangular prism of base $XYZT$ and height δa.)

(ii) Hence show that the pyramid $PABO$ has volume $\frac{1}{6}hbr$.

(iii) Suppose now that $\angle AOB = \frac{2\pi}{n}$ and that n identical pyramids $PABO$ are arranged about O as centre with common vertical axis OP to form the solid C. Show that the volume V_n of C is given by $V_n = \frac{1}{3}r^2 hn \sin\frac{\pi}{n}$.

(iv) Note that when n is large, the solid C approximates a right circular cone. Using the fact that $\frac{\sin x}{x} \to 1$ as $x \to 0$, find $\lim_{n\to\infty} V_n$. Hence verify that a right circular cone of radius r and height h has volume $\frac{1}{3}\pi r^2 h$.

(10) **HSC '91**

(5) (b) A drinking glass having the form of a right circular cylinder of radius a and height h, is filled with water. The glass is slowly tilted over, spilling water out ot it, until it reaches the position where the water's surface bisects the base of the glass. Figure 1 show this position.

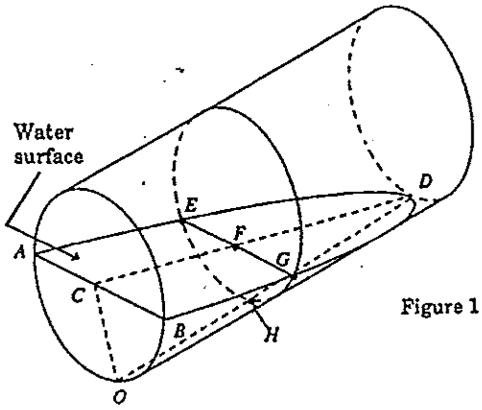


Figure 1

In Figure 1, AB is a diameter of the circular base with centre C, O is the lowest point on the base, and D is the point where the water's surface touches the rim of the glass.
Figure 2 shows a cross-section of the tilted glass parallel to its base. The centre of this circular section is C' and EFG shows the water level. The section cuts the lines CD and OD of Figure 1 in F and H respectively.

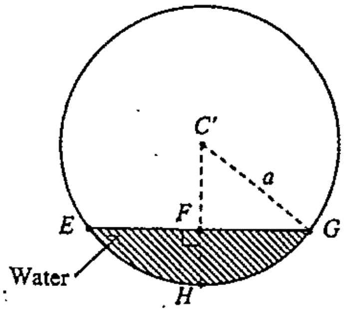


Figure 2

Figure 3 shows the section COD of the tilted glass.

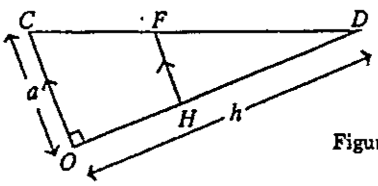


Figure 3

(i) Use Figure 3 to show that $FH = \frac{a}{h}(h - x)$, where $OH = x$.

(ii) Use Figure 2 to show that $C'F = \frac{ax}{h}$ and $\angle HC'G = \cos^{-1}\left(\frac{x}{h}\right)$.

(iii) Use (ii) to show that the area of the shaded segment *EGH* is

$$a^2\left[\cos^{-1}\left(\frac{x}{h}\right)-\left(\frac{x}{h}\right)\sqrt{1-\left(\frac{x}{h}\right)^2}\right].$$

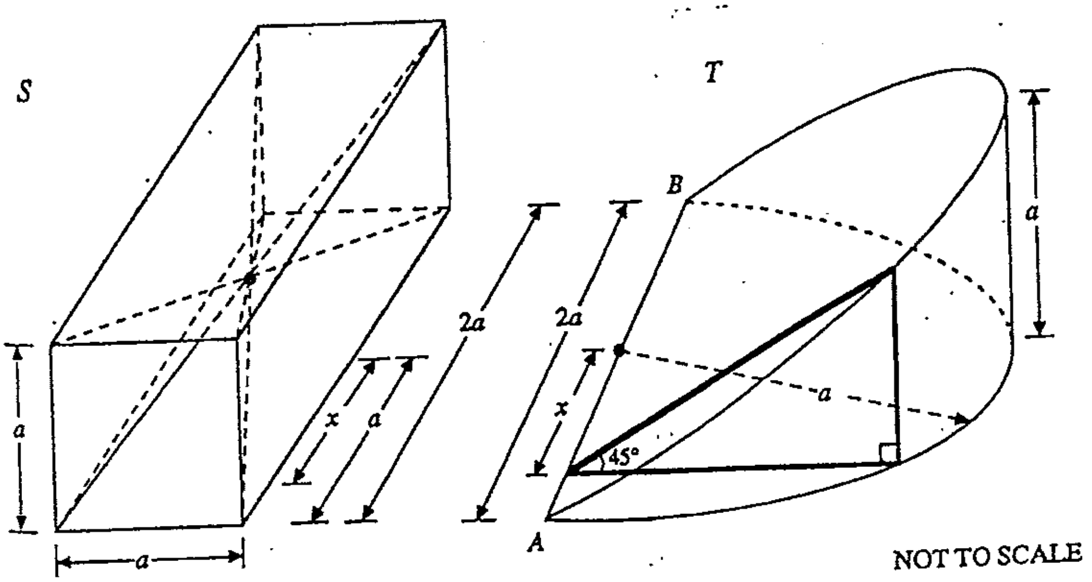
(iv) Given that $\int \cos^{-1}\theta \, d\theta = \theta\cos^{-1}\theta - \sqrt{1-\theta^2}$, find the volume of water in the tilted glass of Figure 1.

$$\boxed{\frac{2a^2h}{3}}$$

HSC '92

(5) (a) The solid S is a rectangular prism of dimensions $a \times a \times 2a$ from which right square pyramids of base $a \times a$ and height a has been removed from each end. The solid T is a wedge that has been obtained by slicing a right circular cylinder of radius a at 45^0 through a diameter AB of its base.

Consider a cross-section of S which is parallel to its square base at distance x from its centre, and a corresponding cross-section of T which is perpendicular to AB and at a distance x from its centre.



(i) The triangular cross-section of T is shown on the diagram.
Show that it has area

$$\frac{1}{2}\left(a^2 - x^2\right).$$

(ii) Draw the cross-section of S and calculate its area.

$\boxed{a^2 - x^2}$

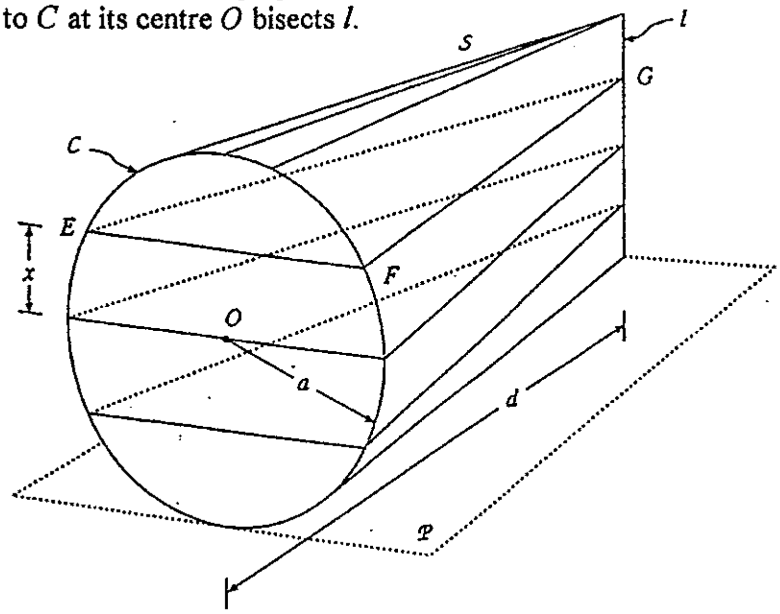
(iii) Express the volumes of S and T as definite integrals

$\boxed{V_s = 2\int_0^a \left(a^2 - x^2\right) dx; \;\; V_T = \int_0^a \left(a^2 - x^2\right) dx}$

(iv) What is the relationship between the volumes of S and T. (There is no need to evaluate either integral.)

HSC '93

(6) (a) The solid S (in the diagram below) is generated by moving a straight edge so that it is always parallel to a fixed plane P. It is constrained to pass through a circle C and line segment l. The circle C, which forms a base for S, has radius a and the line segment l is distance d from C. Both C and l are perpendicular to P and sit on P in such a way that the perpendicular to C at its centre O bisects l.



(i) Calculate the area of the triangular cross-section EFG which is parallel to P and distance x from the centre O of C.

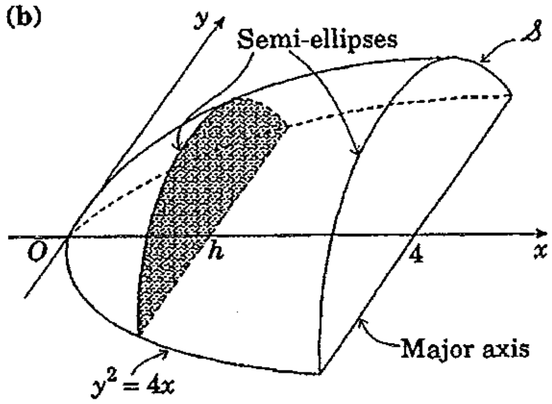
$$d \times \sqrt{a^2 - x^2}$$

(ii) Calculate the volume of S.

$\frac{1}{2}\pi a^2 d$ units^3

HSC 2000
(3)

(b) 6



The base of a solid $\mathcal{S}$ is the region in the x - y plane enclosed by the parabola $y^2 = 4x$ and the line $x = 4$, and each cross-section perpendicular to the x axis is a semi-ellipse with the minor axis one-half of the major axis.

(i) Show that the area of the semi-ellipse at $x = h$ is πh.

(You may assume the area of an ellipse with semi-axes a and b is πab.)

(ii) Find the volume of the solid $\mathcal{S}$.

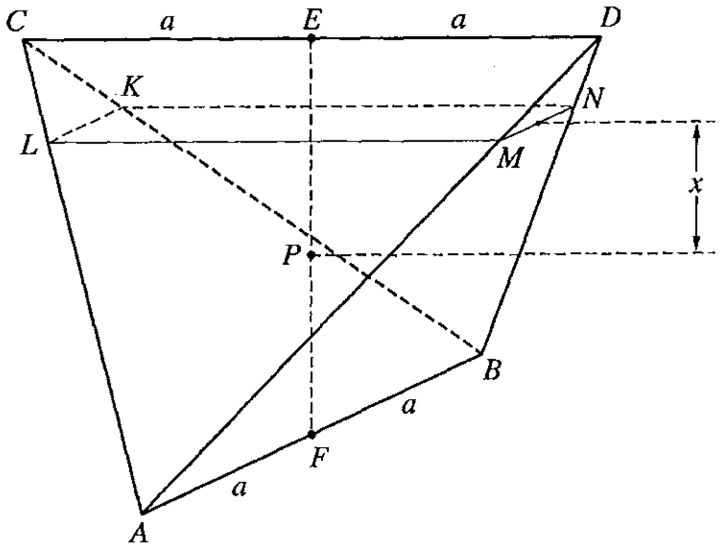
$\boxed{8\pi}$

(iii) Consider the solid $\mathcal{T}$, which is obtained by rotating the region enclosed by the parabola and the line $x = 4$ about the x axis. What is the relation between the volume of $\mathcal{S}$ and the volume of $\mathcal{T}$?

$\boxed{V_{\mathcal{T}} = 4\,V_{\mathcal{S}}}$

HSC 02

(8) (b)



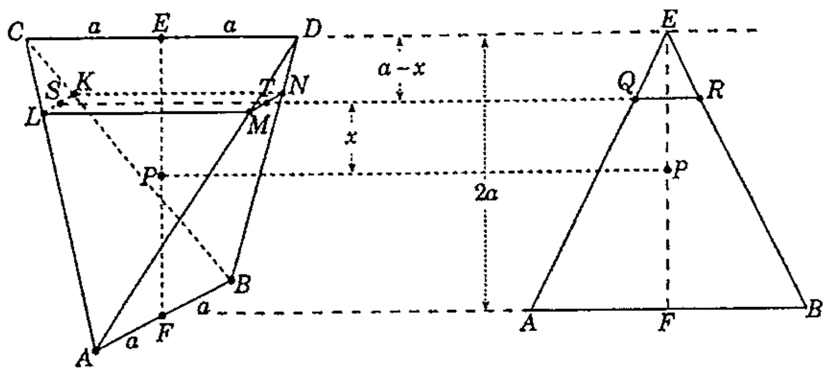
In the diagram, AB and CD are line segments of length $2a$ in horizontal planes at a distance $2a$ apart. The midpoint E of CD is vertically above the midpoint F of AB, and AB lies in the South–North direction, while CD lies in the West–East direction.

The rectangle $KLMN$ is the horizontal cross-section of the tetrahedron $ABCD$ at distance x from the midpoint P of EF (so $PE = PF = a$).

(i) By considering the triangle ABE, deduce that $KL = a - x$, and find the area of the rectangle $KLMN$. **4**

Hint:

(b) Let Q, R, S, T be the midpoints of the sides of rectangle $KLMN$ as shown.



$\boxed{a^2 - x^2}$

(ii) Find the volume of the tetrahedron $ABCD$. **2**

$\boxed{\dfrac{4a^3}{3} \text{ units}^3}$