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YEAR 11 – EXT.1 MATHS

REVIEW TOPIC : AUXILIARY ANGLE METHOD – BOOK 1

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Question 1:

Find all values of θ between $0°$ and $360°$ satisfying the equation

$$3\cos\theta + 4\sin\theta = 1$$

by first expressing $3\cos\theta + 4\sin\theta$ in the form $R\cos(\theta - \alpha)$.

Give your answers to the nearest degree. [6]

$5\cos(\theta - 53.1^0); \theta = 132^0, 335^0$ (to the nearest deg)

Question 2:

Find the value of the acute angle α for which

$$7\cos x - 3\sin x = \sqrt{58}\cos(x+\alpha).$$

for all values of x. [4]

$\boxed{\alpha = 23.2^0 \text{(to 1 d.p.)}}$

Solve each of the following equations, giving your answers correct to one decimal place.

(a) $7 \cos x - 3 \sin x = 5$ $\quad 0 \le x \le 360°$ [5]

$\boxed{x = 25.8^0, 287.8^0 \text{ (to 1 d.p.)}}$

(b) $7 \cos 2x - 3 \sin 2x = 5$ $\quad 0 \le x \le 360°$ [5]

$\boxed{x = 12.9^0, 143.9^0, 192.9^0, 323.9^0 \text{ (to 1 d.p.)}}$

Question 3:

Find all values of x between 0° and 360° which satisfy the equation

$$4\cos x - 3\sin x = 1.$$

Give your answers correct to 1 decimal place. [6]

$x = 41.6^0, 244.7^0$ (to 1 d.p.)

Question 4:

The parametric equations of a curve are

$$x = 4\cos\theta, \quad y = 5\sin\theta, \qquad 0 \le \theta \le 2\pi .$$

(a) Use the parametric equations to calculate $\dfrac{dy}{dx}$ in terms of θ. [3]

$$\boxed{\frac{dy}{dx} = -\frac{5\cos\theta}{4\sin\theta}}$$

(b) Prove that the equation of the tangent to the curve at $(4\cos\theta,\ 5\sin\theta)$ is $5x\cos\theta + 4y\sin\theta = 20$. [3]

(c) If the tangents to the curve pass through the point $(2,\ 5)$, show that

$$\cos\theta + 2\sin\theta = 2.$$ [1]

(d) By solving the equation in (c) find two values of θ (in radians to 2 decimal places) which correspond to the two tangents which pass through the point (2, 5). [5]

$x = 0.64^c, 1.57^c;$ or $36.67^0, 89.95^0$ (to 2 d.p.)

Question 5:

A curve has parametric equations given by

$$x = 2\sin\theta, \quad y = \cos\theta.$$

(a) Find $\frac{dy}{dx}$ in terms of θ. [3]

$$\boxed{\frac{dy}{dx} = -\frac{\sin\theta}{2\cos\theta}}$$

(b) Find the equation of the normal at the point, P, on the curve for which $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$. [3]

$$\boxed{15x - 10y - 18 = 0}$$

(c) Show that the values of θ where this normal meets the curve again are given by $5\cos\theta - 15\sin\theta + 9 = 0$.
Solve this equation for values of θ in the range $0 \le \theta \le 2\pi$ and hence find the co-ordinates of Q, the point where the normal meets the curve again. [6]

$Q(0.56, -0.96)$

SOLUTIONS TO Q1 TO 5:

Question 1:

$$3\cos\theta + 4\sin\theta \equiv R\cos(\theta - \alpha)$$
$$\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

Compare coefficients of $\cos\theta$

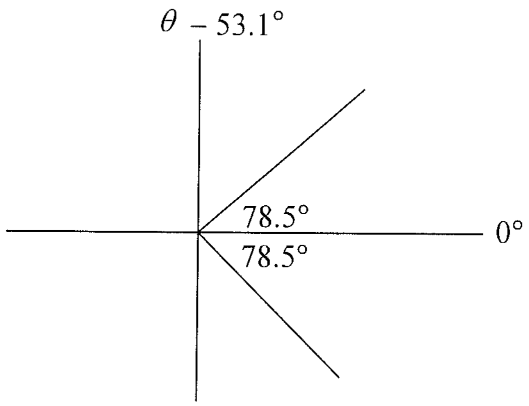
$$3 = R\cos\alpha \qquad -\{1\}$$

Compare coefficients of $\sin\theta$

$$4 = R\sin\alpha \qquad -\{2\}$$

Squaring {1} and {2} and adding gives

$$3^2 + 4^2 = R^2(\cos^2\alpha + \sin^2\alpha)$$
$$= R^2$$
$$\Rightarrow \quad R^2 = 25$$
$$R = \sqrt{25}$$
$$R = 5$$
$$\frac{\{2\}}{\{1\}} \qquad \frac{R\sin\alpha}{R\cos\alpha} = \frac{4}{3}$$
$$\Rightarrow \quad \tan\alpha = \frac{4}{3}$$
$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$
$$\alpha = 53.1°$$
$$\Rightarrow \quad 3\cos\theta + 4\sin\theta \equiv 5\cos(\theta - 53.1°)$$
$$\Rightarrow \quad 5\cos(\theta - 53.1°) = 1$$
$$\Rightarrow \quad \cos(\theta - 53.1°) = \frac{1}{5}$$
$$\cos^{-1}\left(\frac{1}{5}\right) = 78.5°$$



$$\Rightarrow \quad \theta - 53.1° = 78.5° \quad \text{or} \quad \theta - 53.1° = 360° - 78.5°$$
$$\Rightarrow \quad \theta = 131.6° \qquad \theta = 334.6°$$
$$\Rightarrow \quad \theta = 132°,\ 335° \text{ to nearest degree}$$

Question 2:

$$7\cos x - 3\sin x \equiv \sqrt{58}\cos(x+\alpha)$$
$$\equiv \sqrt{58}(\cos x\cos\alpha - \sin x\sin\alpha)$$
$$\equiv \sqrt{58}\cos\alpha\cos x - \sqrt{58}\sin\alpha\sin x$$

Compare coefficients of cos x

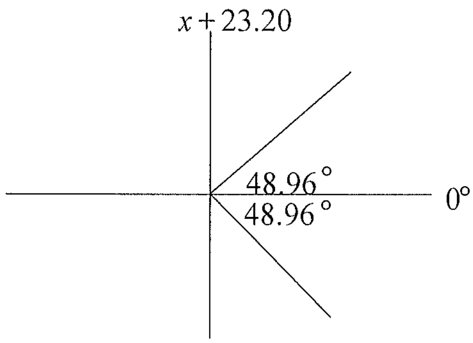
$$7 = \sqrt{58}\cos\alpha \qquad -\{1\}$$

Compare coefficients of sin x

$$-3 = -\sqrt{58}\sin\alpha$$
$$\Rightarrow \quad 3 = \sqrt{58}\sin\alpha \qquad -\{2\}$$
$$\frac{\{2\}}{\{1\}} \quad \frac{\sqrt{58}\sin\alpha}{\sqrt{58}\cos\alpha} = \frac{3}{7}$$
$$\Rightarrow \quad \tan\alpha = \frac{3}{7}$$
$$\alpha = \tan^{-1}\left(\frac{3}{7}\right)$$
$$\alpha = 23.20^\circ \text{ correct to 2 decimal places}$$

(a)

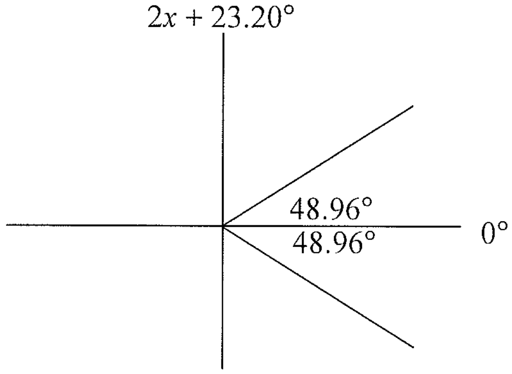
$$7\cos\alpha - 3\sin\alpha = 5$$
$$\Rightarrow \quad \sqrt{58}\cos(x+23.20) = 5$$
$$\cos(x+23.20) = \frac{5}{\sqrt{58}}$$
$$\cos^{-1}\left(\frac{5}{\sqrt{58}}\right) = 48.96^\circ$$



$$\Rightarrow \quad x + 23.20 = 48.96^\circ \quad \text{or} \quad x + 23.20 = 360 - 48.96$$
$$\Rightarrow \quad x = 25.76 \qquad x = 287.84$$
$$\Rightarrow \quad x = 25.8^\circ,\ 287.8^\circ \text{ correct to 1 decimal place}$$

(b) $7\cos 2x - 3\sin 2x = 5$

To solve this equation we can replace x with $2x$ in the quadrant diagram above



$\Rightarrow \quad 2x + 23.20 = 48.96 \quad$ or $\quad 2x + 23.20 = 360 - 48.96$

$\Rightarrow \quad 2x = 25.76 \quad\quad 2x = 287.84$

$\Rightarrow \quad x = 12.88° \quad\quad x = 143.92°$

or $\quad 2x + 23.20 = 360 + 48.96 \quad$ or $\quad 2x + 23.20 = 720 - 48.96$

$\Rightarrow \quad 2x = 385.76 \quad\quad 2x = 647.84$

$\Rightarrow \quad x = 192.88° \quad\quad x = 323.92°$

Solution to the equation is

$x = 12.9°,\ 143.9°,\ 192.9°,\ 323.9°$ correct to 1 decimal place

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Question 3:

$$4\cos x - 3\sin x \equiv R\cos(x + \alpha)$$
$$\equiv R(\cos x\cos\alpha - \sin x\sin\alpha)$$
$$\equiv R\cos\alpha\cos x - R\sin\alpha\sin x$$

Compare coefficients of $\cos x$

$$4 = R\cos\alpha \qquad -\{1\}$$

Compare coefficients of $\sin x$

$$-3 = -R\sin\alpha$$

$\Rightarrow$
$$3 = R\sin\alpha \qquad -\{2\}$$

Squaring {1} and {2} and adding gives

$$4^2 + 3^2 = R^2(\cos^2\alpha + \sin^2\alpha)$$

$\Rightarrow$
$$R^2 = 25$$
$$R = 5$$

$\dfrac{\{2\}}{\{1\}}$
$$\frac{3}{4} = \frac{R\sin\alpha}{R\cos\alpha}$$

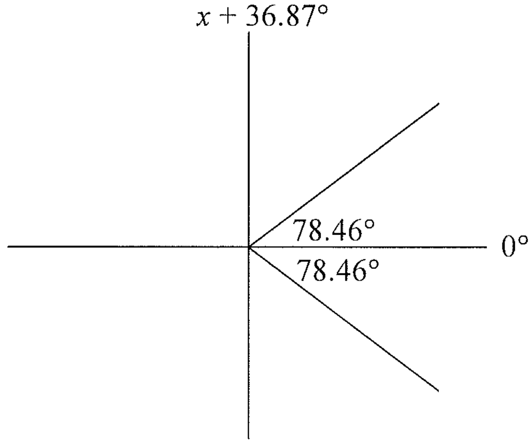
$\Rightarrow$
$$\tan\alpha = 0.75$$
$$\alpha = \tan^{-1}0.75$$
$$= 36.87°$$

$\Rightarrow$
$$4\cos x - 3\sin x = 5\cos(x + 36.87)$$

$\Rightarrow$
$$5\cos(x + 36.87) = 1$$

$$\cos(x + 36.87) = \frac{1}{5}$$

$$\cos^{-1}\left(\frac{1}{5}\right) = 78.46°$$



$\Rightarrow$ $x + 36.87 = 78.46$ or $x + 36.87 = 360-78.46$

$\Rightarrow$ $x = 41.59$ or $x = 244.67$

$x = 41.6°,\ 244.7°$

Question 4:

(a)

$$x = 4\cos\theta \quad , \quad y = 5\sin\theta$$

$$\frac{dx}{d\theta} = -4\sin\theta \qquad \frac{dy}{d\theta} = 5\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= 5\cos\theta \times \frac{1}{-4\sin\theta}$$

$$\frac{dy}{dx} = -\frac{5\cos\theta}{4\sin\theta}$$

(b) The equation of the tangent with gradient m, passing through the point $(x_1,\ y_1)$ is

$$y - y_1 = m(x - x_1)$$

$\Rightarrow$Equation of the tangent passing through $(4\cos\theta,\ 5\sin\theta)$ is

$$y - 5\sin\theta = -\frac{5\cos\theta}{4\sin\theta}(x - 4\cos\theta)$$

$\times\ 4\sin\theta$

$$4y\sin\theta - 20\sin^2\theta = -5\cos\theta(x - 4\cos\theta)$$

$$4y\sin\theta - 20\sin^2\theta = -5x\cos\theta + 20\cos^2\theta$$

$$5x\cos\theta + 4y\sin\theta = 20(\cos^2\theta + \sin^2\theta)$$

$\Rightarrow$ $5x\cos\theta + 4y\sin\theta = 20$

(c) If the tangents pass through the point (2, 5) then substituting $x = 2,\ y = 5$ gives

$$5(2)\cos\theta + 4(5)\sin\theta = 20$$
$$10\cos\theta + 20\sin\theta = 20$$
$$\div 10 \qquad \cos\theta + 2\sin\theta = 2$$

(d)
$$\cos\theta + 2\sin\theta = 2$$

Let $\cos\theta + 2\sin\theta \equiv R\cos(\theta - \alpha)$

Where $R > 0$ and $0 \le \alpha \le \dfrac{\pi}{2}$

$$\cos\theta + 2\sin\theta \equiv R(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$
$$\equiv R\cos\alpha\cos\theta + R\sin\alpha\sin\theta$$

Compare coefficients of $\cos\theta$

$$1 = R\cos\alpha \qquad -\{1\}$$

Compare coefficients of $\sin\theta$

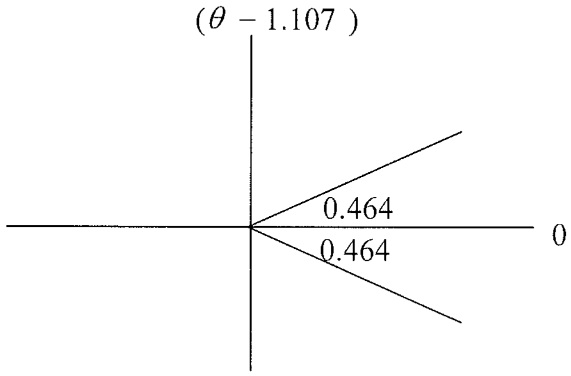
$$2 = R\sin\alpha \qquad -\{2\}$$

Squaring {1} and {2} and adding

$$1^2 + 2^2 = R^2(\cos^2\alpha + \sin^2\alpha)$$
$$\Rightarrow \qquad R^2 = 5$$
$$R = \sqrt{5}$$

$$\frac{\{2\}}{\{1\}} \qquad \frac{R\sin\alpha}{R\cos\alpha} = \frac{2}{1}$$
$$\Rightarrow \qquad \tan\alpha = 2$$
$$\alpha = \tan^{-1} 2$$
$$\alpha = 1.107 \text{ radians}$$

$$\Rightarrow \qquad \cos\theta + 2\sin\theta \equiv \sqrt{5}\cos(\theta - 1.107)$$
$$\Rightarrow \qquad \sqrt{5}\cos(\theta - 1.107) = 2$$
$$\div\sqrt{5} \qquad \cos(\theta - 1.107) = \frac{2}{\sqrt{5}}$$
$$\cos^{-1}\left(\frac{2}{\sqrt{5}}\right) = 0.464 \text{ radians}$$



$\Rightarrow \quad \theta - 1.107 = 0.464$ or $\theta - 1.107 = 2\pi - 0.464$

$\theta = 1.571$ radians $\qquad \theta = 6.926$

This root is outside the interval $0 \le \theta \le 2\pi$.

$\Rightarrow \quad \theta - 1.107 = -0.464$

$\theta = 0.643$ radians

$\Rightarrow$The two values of θ corresponding to the two tangents are
$\theta = 0.64,\ 1.57$ radians correct to 2 decimal places

Question 5:

(a)

$$x = 2\sin\theta, \qquad y = \cos\theta$$

$$\frac{dx}{d\theta} = 2\cos\theta \qquad \frac{dy}{d\theta} = -\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= -\sin\theta \times \frac{1}{2\cos\theta}$$

$$\frac{dy}{dx} = -\frac{\sin\theta}{2\cos\theta}$$

(b) The gradient of the normal $= \dfrac{2\cos\theta}{\sin\theta}$ (using $m_1 m_2 = -1$)

At the point on the curve where $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$ the gradient of the normal.

$$m = \frac{2\left(\frac{3}{5}\right)}{\frac{4}{5}}$$

$$= \frac{3}{2}$$

At this point $x = 2\sin\theta$ $\quad y = \cos\theta$

$$x = 2\times\frac{4}{5} = \frac{8}{5} \qquad y = \frac{3}{5}$$

$\Rightarrow$Equation of the normal, at the point $P\left(\frac{8}{5},\frac{3}{5}\right)$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{3}{5} = \frac{3}{2}\left(x - \frac{8}{5}\right)$$

$\times 10$ $\qquad 10y - 6 = 15x - 24$

$$10y = 15x - 18$$

(c) In order that the normal cuts the curve again substitute $x = 2\sin\theta,\ y = \cos\theta$

$$10\cos\theta = 15\times 2\sin\theta - 18$$

$$10\cos\theta = 30\sin\theta - 18$$

$\div 2$ $\qquad 5\cos\theta - 15\sin\theta + 9 = 0$

Now solve

$$5\cos\theta - 15\sin\theta = -9$$

$$5\cos\theta - 15\sin\theta \equiv R\cos(\theta + \alpha)$$

Where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$

$$5\cos\theta - 15\sin\theta \equiv R(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$

$$\equiv R\cos\alpha\cos\theta - R\sin\alpha\sin\theta$$

Compare coefficients of $\cos\theta$

$$5 = R\cos\alpha \qquad -\{1\}$$

Compare coefficients of $\sin\theta$

$$-15 = -R\sin\alpha$$

$$15 = R\sin\alpha \qquad -\{2\}$$

Squaring {1} and {2} and adding gives

$$5^2 + 15^2 = R^2(\cos^2\alpha + \sin^2\alpha)$$

$\Rightarrow$ $\qquad R^2 = 250$

$$R = \sqrt{250}$$

$\frac{\{2\}}{\{1\}} =$ $\qquad \frac{R\sin\alpha}{R\cos\alpha} = \frac{15}{5}$

$\Rightarrow$ $\qquad \tan\alpha = 3$

$$\alpha = \tan^{-1}3$$

$$\alpha = 1.249 \text{ radians}$$

$\Rightarrow$ $\qquad 5\cos\theta - 15\sin\theta \equiv \sqrt{250}\cos(\theta + 1.249)$

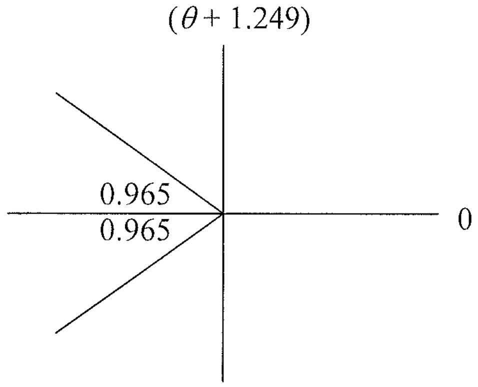
The equation becomes

$$\sqrt{250}\cos(\theta + 1.249) = -9$$

$$\cos(\theta + 1.249) = \frac{-9}{\sqrt{250}}$$

In the quadrant diagram, the acute angle the directions make with the x axis is

$$\cos^{-1}\left(\frac{9}{\sqrt{250}}\right) = 2.17 \text{ radians}$$



$\Rightarrow$ $\theta + 1.249 = \pi - 0.965$ or $\theta + 1.249 = \pi + 0.965$

$\theta = 0.928$ or $\theta = 2.858$

The solution $\theta = 0.928$ is the value of θ at the point P $\left(\sin^{-1}\frac{4}{5} = 0.93\right)$

$\Rightarrow \theta = 2.858$ is the value of θ at the point Q, the other point where the normal cuts the curve.

Co-ordinates of Q (2sin2.858, cos2.858) = (0.56, −0.96)