C.E.M. TUITION

Student Name : ____________________

Review Topic : Binomial Theorem

(HSC)

Year 12 - 3 Unit

1. **Find the coefficient of x^{17} in the expansion of $\left(x^2 - 2x\right)^{12}$.**

2. **Find the middle term of $\left(x^2 - \frac{1}{x}\right)^{16}$.**

3. Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^{12}.$$

4. **Determine the coefficient of x^9 in the expansion of**

$(1+x-x^3)(1-x^2)^9$.

5. Expand $\left(1+x+2x^2\right)^{10}$ in ascending powers of x as far as the term x^4.

6. Determine the coefficient of x^8 in the expansion of

$$\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^6.$$

7. (a) Factorise $1+x+x^2+x^3$

(b) Determine the coefficient of x^4 in the expansion of

$(1+x+x^2+x^3)^3$.

8. Find the numerically greatest coefficient in the expansion of $(2-3x)^{10}$. Also find the greatest term when $x=-2$.

9. (i) State the binomial expansion of $(1+x)^n$ and hence by considering the coefficients of x^r in the expansions of $(1+x)^{n+1}$ and $(1+x)(1+x)^n$) prove the Pascal triangle relation ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(ii) By evaluating $\int_0^2 (1-x)^{2n-1}\,dx$

in 2 different ways, prove that the following identity holds for all odd values of m:

$$2-\frac{2^2}{2}\,{}^mC_1+\frac{2^3}{3}\,{}^mC_2-\ \dots\ +(-1)^r\cdot\frac{2^{r+1}}{r+1}\cdot{}^mC_r$$
$$+\ \dots\ +(-1)^m\cdot\frac{2^{m+1}}{m+1}\cdot{}^mC_m=0$$

10. Using the expansion $(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$, prove the following:

(i) $1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ \dots\ + (-1)^n\, {}^nC_n = 0$

(ii) $1 - \frac{1}{2}\, {}^nC_1 + \frac{1}{3}\, {}^nC_2 - \ \dots\ + \frac{(-1)^n}{n+1}\, {}^nC_n = \frac{1}{n+1}$

1. $\left(x^2-2x\right)^{12}$, $n=12$,
$a=x^2,\ b=-2x$
$T_{r+1}={}^nC_r a^{n-r}b^r$
$={}^{12}C_r\left(x^2\right)^{12-r}(-2x)^r$
$={}^{12}C_r\cdot(-2)^{12-r}\cdot x^{24-r}$
For the term containing x^{17},
$24-r=17,\ r=7$
$T_8={}^{12}C_7\cdot(-2)^7\cdot x^{17}$
The required coefficient is ${}^{12}C_7\cdot(-2)^7=-101\,376$

2. $\left(x^2-\frac{1}{x}\right)^{16}$, $n=16$
$a=x^2,\ b=-\frac{1}{x}$
There are 17 terms.
9th term is the middle term, so $r=8$.
$T_9={}^{16}C_8\cdot\left(x^2\right)^8\cdot\left(-\frac{1}{x}\right)^8$
$={}^{16}C_8x^8$ or $12\,870x^8$

3. $\left(2x^2+\frac{1}{x}\right)^{12}$, $a=2x^2$
$b=\frac{1}{x},\ n=12$
$T_{r+1}={}^nC_r a^{n-r}b^r$
$={}^{12}C_r\cdot\left(2x^2\right)^{12-r}\cdot\left(\frac{1}{x}\right)^r$
$={}^{12}C_r\cdot(2)^{12-r}\cdot x^{24-3r}$
For the constant term,
$24-3r=0,\ r=8$
$T_9={}^{12}C_8\cdot 2^4=7920.$

4. Let $E=\left(1+x-x^3\right)\left(1-x^2\right)^9$.
${}^9C_1=9,\ {}^9C_2=36,\ {}^9C_3=84,$
${}^9C_4=126={}^9C_5.$
$E=\left(1+x-x^3\right)\left(1-9x^2+36x^4-84x^6+126x^8-\ldots\right)$
The coefficient of x^9 is
$84+126=210.$

5. Let $E=\left(1+x+2x^2\right)^{10}$
and $y=x+2x^2$
Then $E=(1+y)^{10}$
${}^{10}C_1=10,\ {}^{10}C_2=45,$
${}^{10}C_3=120,\ {}^{10}C_4=210$
$y^2=x^2+4x^3+4x^4$
$y^3=\left(x+2x^2\right)^3$
$=x^3+6x^4+\ldots$
$y^4=\left(x+2x^2\right)^4=x^4+\ldots$
$E=1+10y+45y^2+120y^3+210y^4+\ldots$
$=1+10\left(x+2x^2\right)+45\left(x^2+4x^3+4x^4\right)+120\left(x^3+6x^4+\ldots\right)+210\left(x^4+\ldots\right)$
$\therefore\ E=1+10x+65x^2+300x^3+1110x^4$ + higher powers of x.

6. Let $E=\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^6$
Using $a^mb^m=(ab)^m$,
$E=\left(x^2-\frac{1}{x^2}\right)^6$
$T_{r+1}={}^6C_r\left(x^2\right)^{6-r}\left(\frac{-1}{x^2}\right)^r$
$={}^6C_r\cdot(-1)^r\cdot x^{12-4r}$
For the term containing x^7,
$12-4r=8,\ r=1.$
$T_2={}^6C_1x^8=6x^8$
The required coefficient is 6.

7. (a) $1+x+x^2+x^3$
$=(1+x)\left(1+x^2\right)$

(b) Let $E=\left(1+x+x^2+x^3\right)^3$
$\therefore\ E=(1+x)^3\left(1+x^2\right)^3$
$=\left(1+3x+3x^3+x^3\right)\times\left(1+3x^2+3x^4+x^6\right)$
The coefficient of
$x^4=3+9$
$=12$

8. $(2-3x)^{10}$, $n=10$,
$a=2,\ b=-3x$
We first prove:
${}^nC_r:{}^nC_{r-1}=(n-r+1):r$
${}^nC_r$
$=\frac{n(n-1)\ldots(n-r)(n-r+1)}{1\cdot2\ldots(r-1)\cdot r}$
$={}^nC_{r-1}\cdot\frac{n-r+1}{r}$
$\therefore\ {}^nC_r:{}^nC_{r-1}=(n-r+1):r$
We have:
$T_{r+1}={}^nC_ra^{n-r}b^r$
$T_r={}^nC_{r-1}a^{n-r+1}b^{r-1}$
Using the result proved above:
$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\cdot\frac{b}{a}$
Substituting,
$\frac{T_{r+1}}{T_r}=\frac{11-r}{r}\cdot\left(\frac{-3x}{2}\right)$
We are only interested in the numerical value of the coefficient, so
coef. of $T_{r+1}\ge$ coef. of T_r,
if $\frac{11-r}{r}\cdot\left(\frac{3}{2}\right)\ge1\Rightarrow r\le6.6$
So for $r=0,\ 1,\ \ldots,\ 6$
$T_{r+1}\ge T_r$ and for $7\le r\le10$,
$T_r\ge T_{r+1}$.
Hence T_7 has the greatest coefficient
$={}^{10}C_6\cdot2^4\cdot(-3)^6$
$=2\,449\,440$
When $x=-2$, using the result proved above:
$\frac{T_{r+1}}{T_r}=\frac{11-r}{r}\cdot\left(\frac{-3x}{2}\right)$
$=\frac{3(11-r)}{r}$
$T_{r+1}\ge T_r$ if $33-3r\ge r$
$r\le8.25$
$\therefore$ The greatest term occurs when $r=8$.
Then $T_9={}^{10}C_8\cdot2^2\cdot6^8$

9. (i) $(1+x)^n$

$= 1 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_r x^r + {}^nC_n x^n.$

Now,

$(1+x)^{n+1}$

$= (1+x)(1+x)^n$

$= (1+x)\left(1 + {}^nC_1 x + \ldots + {}^nC_{r-1} x^{r-1} + {}^nC_r x^r + \ldots + {}^nC_n x^n\right)$

Comparing the coefficient of x^r on both sides:

${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$

(ii) Let $m = 2n+1$, then:

$(1-x)^{2n+1}$

$= (1-x)^m$

$= 1 - {}^mC_1 x + {}^mC_2 x^2 + \ldots + (-1)^r \cdot {}^mC_r x^r + \ldots + (-1)^m \cdot {}^mC_m x^m.$

We integrate both sides with respect to x:

L.H.S.

$= \int_0^2 (1-x)^m \, dx$

$= -\left[\frac{(1-x)^{m+1}}{m+1}\right]_0^2$

$= \frac{-1}{m+1}\left[(-1)^{m+1} - 1\right]$

Now $(-1)^{m+1} = (-1)^{2n+2} = 1$

Hence L.H.S. = 0.

R.H.S.

$= \left[x - \frac{x^2}{2} \cdot {}^mC_1 + \ldots + (-1)^r \frac{x^{r+1}}{r+1} + \ldots\right]_0^2$

$= 2 - \frac{2^2}{2} \cdot {}^mC_1 + \frac{2^3}{3} \cdot {}^mC_2 - \ldots + (-1)^m \cdot \frac{2^{m+1}}{m+1} \cdot {}^mC_m$

But L.H.S. = 0, hence

$2 - \frac{2^2}{2} \cdot {}^mC_1 + \ldots + (-1)^r \cdot {}^mC_r \cdot \frac{2^{r+1}}{r+1} + \ldots + (-1)^m \cdot {}^mC_m \cdot \frac{2^{m+1}}{m+1} = 0.$

10. $(1+x)^n = 1 + {}^nC_1 x + {}^nC_1 x^2 + \ldots + {}^nC_r x^r + \ldots + {}^nC_n x^n \quad \ldots (1)$

(i) Put $x = -1$, then:

$0 = 1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n \cdot {}^nC_n$

Hence the result.

(ii) Integrate both sides of (1) with respect to x:

$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^0$

$= \left[x + {}^nC_1 \frac{x^2}{2} + \ldots + {}^nC_n \frac{x^{n+1}}{n+1}\right]_{-1}^0$

L.H.S. $= \frac{1}{n+1}$

R.H.S. $= 0 - \left(-1 + \frac{1}{2}{}^nC_1 - \frac{1}{3}{}^nC_2 + \ldots + (-1)^n \, {}^nC_n\right)$

$\therefore\ 1 - \frac{1}{2}{}^nC_1 + \frac{1}{3}{}^nC_2 - \ldots + \frac{(-1)^n}{n+1}{}^nC_n = 0.$