C.E.M. TUITION

Student Name : ______________________________

Review Topic : Binomial Theorem

(HSC)

Year 12 - 3 Unit

1. Find the coefficient of x^{17} in the expansion of $\left(x^2 - 2x\right)^{12}$.

2. Find the middle term of $\left(x^2 - \frac{1}{x}\right)^{16}$.

3. **Find the term independent of x in the expansion of**

$$\left(2x^2+\frac{1}{x}\right)^{12}.$$

4. **Determine the coefficient of x^9 in the expansion of**

$(1+x-x^3)(1-x^2)^9$.

5. Expand $\left(1+x+2x^2\right)^{10}$ in ascending powers of x as far as the term x^4.

6. Determine the coefficient of x^8 in the expansion of

$$\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^6.$$

7. (a) Factorise $1+x+x^2+x^3$

(b) Determine the coefficient of x^4 in the expansion of $\left(1+x+x^2+x^3\right)^3$.

8. Find the numerically greatest coefficient in the expansion of $(2-3x)^{10}$. Also find the greatest term when $x = -2$.

9. (i) State the binomial expansion of $(1+x)^n$ and hence by considering the coefficients of x^r in the expansions of $(1+x)^{n+1}$ and $(1+x)(1+x)^n$) prove the Pascal triangle relation ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(ii) By evaluating $\int_0^2 (1-x)^{2n-1}\, dx$

in 2 different ways, prove that the following identity holds for all odd values of m:

$$2-\frac{2^2}{2}\,{}^mC_1+\frac{2^3}{3}\,{}^mC_2-\ \dots\ +(-1)^r\cdot\frac{2^{r+1}}{r+1}\cdot{}^mC_r$$

$$+\ \dots\ +(-1)^m\cdot\frac{2^{m+1}}{m+1}\cdot{}^mC_m=0$$

10. Using the expansion $(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$, prove the following:

(i) $1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ \dots\ + (-1)^n\, {}^nC_n = 0$

(ii) $1 - \frac{1}{2}\,{}^nC_1 + \frac{1}{3}\,{}^nC_2 - \ \dots\ + \frac{(-1)^n}{n+1}\,{}^nC_n = \frac{1}{n+1}$

1. $\left(x^2-2x\right)^{12}$, $n=12$,

$a=x^2,\ b=-2x$

$T_{r+1}={}^nC_r a^{n-r}b^r$

$={}^{12}C_r\left(x^2\right)^{12-r}(-2x)^r$

$={}^{12}C_r\cdot(-2)^{12-r}\cdot x^{24-r}$

For the term containing x^{17},

$24-r=17,\ r=7$

$T_8={}^{12}C_7\cdot(-2)^7\cdot x^{17}$

The required coefficient is ${}^{12}C_7\cdot(-2)^7=-101376$

2. $\left(x^2-\frac{1}{x}\right)^{16}$, $n=16$

$a=x^2,\ b=-\frac{1}{x}$

There are 17 terms. 9th term is the middle term, so $r=8$.

$T_9={}^{16}C_8\cdot\left(x^2\right)^8\cdot\left(-\frac{1}{x}\right)^8$

$={}^{16}C_8x^8$ or $12\,870x^8$

3. $\left(2x^2+\frac{1}{x}\right)^{12}$, $a=2x^2$

$b=\frac{1}{x},\ n=12$

$T_{r+1}={}^nC_r a^{n-r}b^r$

$={}^{12}C_r\cdot\left(2x^2\right)^{12-r}\cdot\left(\frac{1}{x}\right)^r$

$={}^{12}C_r\cdot(2)^{12-r}\cdot x^{24-3r}$

For the constant term,

$24-3r=0,\ r=8$

$T_9={}^{12}C_8\cdot 2^4=7920.$

4. Let $E=\left(1+x-x^3\right)\left(1-x^2\right)^9$.

${}^9C_1=9,\ {}^9C_2=36,\ {}^9C_3=84,$

${}^9C_4=126={}^9C_5.$

$E=\left(1+x-x^3\right)\left(1-9x^2+36x^4-84x^6+126x^8-\ldots\right)$

The coefficient of x^9 is $84+126=210.$

5. Let $E=\left(1+x+2x^2\right)^{10}$

and $y=x+2x^2$

Then $E=(1+y)^{10}$

${}^{10}C_1=10,\ {}^{10}C_2=45,$

${}^{10}C_3=120,\ {}^{10}C_4=210$

$y^2=x^2+4x^3+4x^4$

$y^3=\left(x+2x^2\right)^3$

$=x^3+6x^4+\ldots$

$y^4=\left(x+2x^2\right)^4=x^4+\ldots$

$E=1+10y+45y^2+120y^3+210y^4+\ldots$

$=1+10\left(x+2x^2\right)+45\left(x^2+4x^3+4x^4\right)+120\left(x^3+6x^4+\ldots\right)+210\left(x^4+\ldots\right)$

$\therefore\ E=1+10x+65x^2+300x^3+1110x^4$ + higher powers of x.

6. Let $E=\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^6$

Using $a^mb^m=(ab)^m$,

$E=\left(x^2-\frac{1}{x^2}\right)^6$

$T_{r+1}={}^6C_r\left(x^2\right)^{6-r}\left(\frac{-1}{x^2}\right)^r$

$={}^6C_r\cdot(-1)^r\cdot x^{12-4r}$

For the term containing x^7,

$12-4r=8,\ r=1.$

$T_2={}^6C_1x^8=6x^8$

The required coefficient is 6.

7. (a) $1+x+x^2+x^3$

$=(1+x)\left(1+x^2\right)$

(b) Let $E=\left(1+x+x^2+x^3\right)^3$

$\therefore\ E=(1+x)^3\left(1+x^2\right)^3$

$=\left(1+3x+3x^3+x^3\right)\times\left(1+3x^2+3x^4+x^6\right)$

The coefficient of

$x^4=3+9$

$=12$

8. $(2-3x)^{10}$, $n=10$,

$a=2,\ b=-3x$

We first prove:

${}^nC_r:{}^nC_{r-1}=(n-r+1):r$

${}^nC_r$

$=\frac{n(n-1)\ldots(n-r)(n-r+1)}{1\cdot2\ldots(r-1)\cdot r}$

$={}^nC_{r-1}\cdot\frac{n-r+1}{r}$

$\therefore\ {}^nC_r:{}^nC_{r-1}=(n-r+1):r$

We have:

$T_{r+1}={}^nC_ra^{n-r}b^r$

$T_r={}^nC_{r-1}a^{n-r+1}b^{r-1}$

Using the result proved above:

$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\cdot\frac{b}{a}$

Substituting,

$\frac{T_{r+1}}{T_r}=\frac{11-r}{r}\cdot\left(\frac{-3x}{2}\right)$

We are only interested in the numerical value of the coefficient, so

coef. of $T_{r+1}\ge$ coef. of T_r,

if $\frac{11-r}{r}\cdot\left(\frac{3}{2}\right)\ge1\Rightarrow r\le6.6$

So for $r=0,\ 1,\ \ldots,\ 6$

$T_{r+1}\ge T_r$ and for $7\le r\le10$, $T_r\ge T_{r+1}$.

Hence T_7 has the greatest coefficient

$={}^{10}C_6\cdot2^4\cdot(-3)^6$

$=2\,449\,440$

When $x=-2$, using the result proved above:

$\frac{T_{r+1}}{T_r}=\frac{11-r}{r}\cdot\left(\frac{-3x}{2}\right)$

$=\frac{3(11-r)}{r}$

$T_{r+1}\ge T_r$ if $33-3r\ge r$

$r\le8.25$

$\therefore$ The greatest term occurs when $r=8$.

Then $T_9={}^{10}C_8\cdot2^2\cdot6^8$

9. (i) $(1+x)^n$

$= 1 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + {}^nC_nx^n.$

Now,

$(1+x)^{n+1}$

$= (1+x)(1+x)^n$

$= (1+x)\left(1 + {}^nC_1x + \ldots + {}^nC_{r-1}x^{r-1} + {}^nC_rx^r + \ldots + {}^nC_nx^n\right)$

Comparing the coefficient of x^r on both sides:

${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$

(ii) Let $m = 2n+1$, then:

$(1-x)^{2n+1}$

$= (1-x)^m$

$= 1 - {}^mC_1x + {}^mC_2x^2 + \ldots + (-1)^r \cdot {}^mC_rx^r + \ldots + (-1)^m \cdot {}^mC_mx^m.$

We integrate both sides with respect to x:

L.H.S.

$= \int_0^2 (1-x)^m \, dx$

$= -\left[\frac{(1-x)^{m+1}}{m+1}\right]_0^2$

$= \frac{-1}{m+1}\left[(-1)^{m+1} - 1\right]$

Now $(-1)^{m+1} = (-1)^{2n+2} = 1$

Hence L.H.S. = 0.

R.H.S.

$= \left[x - \frac{x^2}{2} \cdot {}^mC_1 + \ldots + (-1)^r \frac{x^{r+1}}{r+1} + \ldots\right]_0^2$

$= 2 - \frac{2^2}{2} \cdot {}^mC_1 + \frac{2^3}{3} \cdot {}^mC_2 - \ldots + (-1)^m \cdot \frac{2^{m+1}}{m+1} \cdot {}^mC_m$

But L.H.S. = 0, hence

$2 - \frac{2^2}{2} \cdot {}^mC_1 + \ldots + (-1)^r \cdot {}^mC_r \cdot \frac{2^{r+1}}{r+1} + \ldots + (-1)^m \cdot {}^mC_m \cdot \frac{2^{m+1}}{m+1} = 0.$

10. $(1+x)^n = 1 + {}^nC_1x + {}^nC_1x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n \quad \ldots (1)$

(i) Put $x = -1$, then:

$0 = 1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n \cdot {}^nC_n$

Hence the result.

(ii) Integrate both sides of (1) with respect to x:

$\left[\frac{(1+x)^{n+1}}{n+1}\right]_{-1}^0$

$= \left[x + {}^nC_1\frac{x^2}{2} + \ldots + {}^nC_n\frac{x^{n+1}}{n+1}\right]_{-1}^0$

L.H.S. $= \frac{1}{n+1}$

R.H.S. $= 0 - \left(-1 + \frac{1}{2}{}^nC_1 - \frac{1}{3}{}^nC_2 + \ldots + (-1)^n\, {}^nC_n\right)$

$\therefore\ 1 - \frac{1}{2}{}^nC_1 + \frac{1}{3}{}^nC_2 - \ldots + \frac{(-1)^n}{n+1}\, {}^nC_n = 0.$

5. Expand $\left((1+x)+(2x^2)\right)^{10}$ in ascending powers of x as far as the term x^4.

$^{10}C_0 (1+x)^{10}$

$(1+x+2x^2)^{10} = {}^{10}C_1 (1+x)^9 (2x^2) + {}^{10}C_2 (1+x)^8 (2x^2)^2 + \ldots$

$(1 + {}^{10}C_1 x + {}^{10}C_2 x^2 + {}^{10}C_3 x^3 + {}^{10}C_4 x^4 + \ldots)$

$= {}^{10}C_1 (2x^2)(1 + {}^9C_1 x + {}^9C_2 x^2 + \ldots) + {}^{10}C_2\, 4x^2 (1 + {}^8C_1 x + {}^8C_2 x^2 + \ldots)$

$1 + 10x + 45x^2 + 120x^3 + 210x^4 + \ldots$

$= 20x^2(1 + 9x + 36x^2 + \ldots) + 180x^2(1 + 8x + 28x^2 + \ldots)$

$1 + 10x + 45x^2 + 120x^3 + 210x^4 +$

$= 20x^2 + 180x^3 + 720x^4 + 180x^2 + 1440x^3 + 5040x^4 \ldots$

$= 1 + 10x + 245x^2 + 1740x^3 + 5970x^4$

$[1 + (x + 2x^2)]^{10}$

$= {}^{10}C_0 + {}^{10}C_1 (x+2x^2)^1 + {}^{10}C_2 (x+2x^2)^2 + {}^{10}C_3 (x+2x^2)^3 + {}^{10}C_4 (x+2x^2)^4 + \ldots$

$= 1 + 10x + 20x^2 + 45(x^2 + 4x^3 + 4x^4) + 120(x^3 + 3(x^2)(2x^2) + \ldots$

$+ 210[1.x^4 + \ldots]$

$= 1 + 10x + 65x^2 + 300x^3 + 1110x^4 + \ldots$

1 3 3 1

1 4 6 4 1