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YEAR 12 – EXT.1 MATHS

REVIEW TOPIC : HARDER 2U PROBLEMS – SP1

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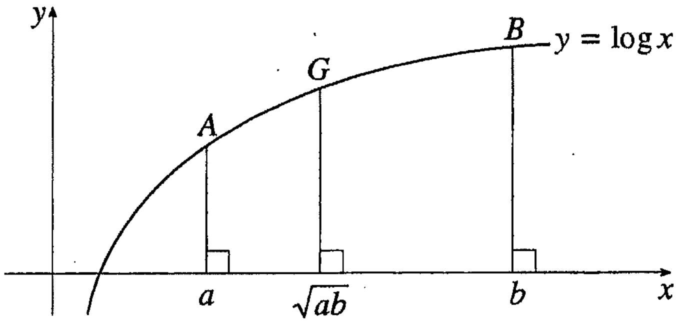
EXERCISES:

(1) (a)

Determine: $\lim_{h \to 0} \frac{\sin \frac{h}{2}}{h}$

(b)



M is the midpoint of AB. Show that GM is parallel to the x axis.

(2)

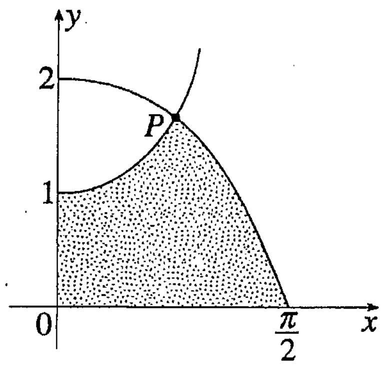
P is a point where the graphs of $y = \tan x$ and $y = \cos x$ intersect.

(i) If the x coordinate of P is α , show that $\sin\alpha = \cos^2\alpha$.

(ii) Prove that, at all such points P, the tangents to the two graphs are perpendicular.

(3)

P is the point of intersection between $x = 0$ and $x = \dfrac{\pi}{2}$ of the graphs of $y = \sec x$ and $y = 2\cos x$, as shown.



(i) Verify that the x coordinate of P is $\dfrac{\pi}{4}$.

(ii) The shaded region makes a revolution about the x axis.

Show that the volume of the resulting solid is $\dfrac{\pi^2}{2}$ cubic units.

(4)

(i) Sketch a graph of $y = |2x-3|$.

(ii) Show graphically that the equation $|2x-3| = x-2$ has no solutions.

(5)

$f(x) = \sin^3 x + \cos^3 x$

(i) Show that, when $f(x) = 0$, $\tan x = -1$.

(ii) Show that, when $f'(x) = 0$, $\sin x = 0$, $\cos x = 0$ or $\tan x = 1$.

(iii) Sketch a graph of $y = f(x)$ for $0 \le x \le 2\pi$.

(6)

A solid cylinder, radius r and height h, is to be constructed under the condition that the sum of its height and circumference is S, where S is constant.

Prove that

(i) if it is given its maximum possible volume, $h = \frac{S}{3}$

(ii) if it is given its maximum possible total surface area, $r + h = \frac{S}{2}$.

SOLUTIONS:

(1) (a)

$$\lim_{h \to 0} \frac{\sin \frac{h}{2}}{h} = \lim_{h \to 0} \frac{1}{2} \cdot \frac{\sin \frac{h}{2}}{\frac{h}{2}}$$
$$= \frac{1}{2}$$

(b)

$A(a, \log a)$, $B(b, \log b)$

At M, $y = \dfrac{\log a + \log b}{2}$

At G,
$$y = \log \sqrt{ab}$$
$$= \log (ab)^{\frac{1}{2}}$$
$$= \frac{1}{2}(\log ab)$$
$$= \frac{\log a + \log b}{2}$$

Hence GM is parallel to the x axis.

(2)

(i)
$$\tan \alpha = \cos \alpha$$
$$\frac{\sin \alpha}{\cos \alpha} = \cos \alpha$$
$$\sin \alpha = \cos^2 \alpha$$

(ii)
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

Product of gradients
$$= (\sec^2 \alpha)(-\sin \alpha)$$
$$= (\sec^2 \alpha)(-\cos^2 \alpha), \text{ from part (i)}$$
$$= -1$$

(3)

(i) $2\cos\frac{\pi}{4} = \frac{2}{\sqrt{2}}$
$= \sqrt{2}$
$= \sec\frac{\pi}{4}$

(ii) For the region above $0 < x < \frac{\pi}{4}$,

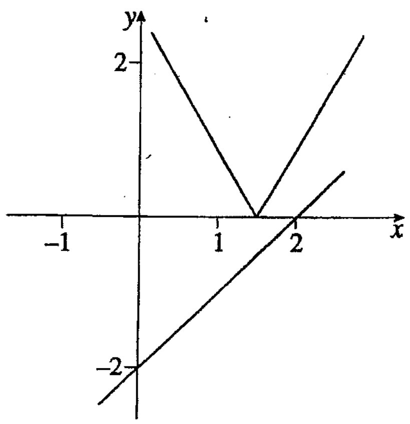
$$V = \pi\int_0^{\frac{\pi}{4}} \sec^2 x\ dx$$
$$= \pi\Big[\tan x\Big]_0^{\frac{\pi}{4}}$$
$$= \pi \text{ cubic units.}$$

For the region above $\frac{\pi}{4} < x < \frac{\pi}{2}$,

$$V = \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos^2 x\ dx$$
$$= 2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos 2x)\ dx$$
$$= 2\pi\left[x + \frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= 2\pi\left[\frac{\pi}{2} + 0 - \frac{\pi}{4} - \frac{1}{2}\right]$$
$$= \left(\frac{\pi^2}{2} - \pi\right) \text{ cubic units.}$$

Hence total volume is $\frac{\pi^2}{2}$ cubic units.

(4) (i)



(5)

(i) $\sin^3 x + \cos^3 x = 0$
$\tan^3 x + 1 = 0$
$\tan^3 x = -1$
$\tan x = -1$

(ii) $3\sin^2 x\cos x - 3\cos^2 x\sin x = 0$

$3\sin x\cos x(\sin x - \cos x) = 0$

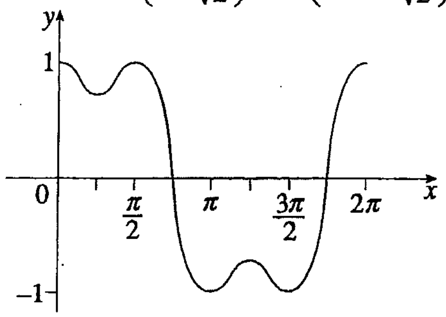
$\therefore\ \sin x = 0,\ \cos x = 0,$ or $\sin x = \cos x$

$\tan x = 1$

(iii) $\sin x = 0$ at $(0,1)$, $(\pi,-1)$ and $(2\pi,1)$

$\cos x = 0$ at $\left(\frac{\pi}{2},1\right)$ and $\left(\frac{3\pi}{2},-1\right)$

$\tan x = 1$ at $\left(\frac{\pi}{4},\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4},-\frac{1}{\sqrt{2}}\right)$



(6)

$2\pi r + h = S$

(i) $V = \pi r^2(S - 2\pi r)$

$= \pi\left(Sr^2 - 2\pi r^3\right)$

$\frac{dV}{dr} = \pi\left(2Sr - 6\pi r^2\right)$

$= 2\pi r(S - 3\pi r)$

$\frac{d^2V}{dr^2} = 2\pi(S - 6\pi r)$

When $r = \frac{S}{3\pi}$, $h = \frac{S}{3}$,

$\frac{dV}{dr} = 0$, $\frac{d^2V}{dr^2} < 0$,

and hence V is maximum.

(ii) $A = 2\pi r^2 + 2\pi r(S - 2\pi r)$

$= 2\pi\left(r^2 + Sr - 2\pi r^2\right)$

$= 2\pi\left(Sr + r^2 - 2\pi r^2\right)$

$\frac{dA}{dr} = 2\pi(S + 2r - 4\pi r)$

When $\frac{dA}{dr} = 0,\quad S = 4\pi r - 2r$

$$
\begin{aligned}
&= r(4\pi - 2) \\
r &= \frac{S}{4\pi - 2} \\
h &= S - \frac{2\pi S}{4\pi - 2} \\
&= \frac{2\pi S - 2S}{4\pi - 2} \\
r + h &= \frac{2\pi S - S}{4\pi - 2} \\
&= \frac{S(2\pi - 1)}{2(2\pi - 1)} \\
&= \frac{S}{2}
\end{aligned}
$$

$\frac{d^2A}{dr^2} = 2\pi(2 - 4\pi) < 0,$ hence maximum.