NAME :

Centre of Excellence in Mathematics
S201 / 414 GARDENERS RD. ROSEBERY 2018
www.cemtuition.com.au

MOBILE 0412 880 475

PHONE 96663331

YEAR 12 – EXT.1 MATHS

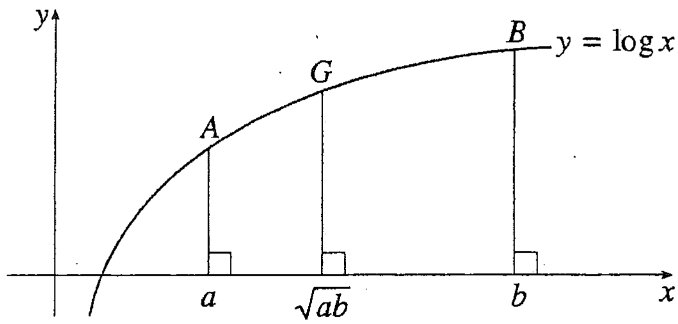
REVIEW TOPIC (SP2)
HARDER 2U PROBLEMS

EXERCISES:

(1) (a)

Determine: $\lim_{h \to 0} \dfrac{\sin \frac{h}{2}}{h}$

(b)



M is the midpoint of AB. Show that GM is parallel to the x axis.

(2)

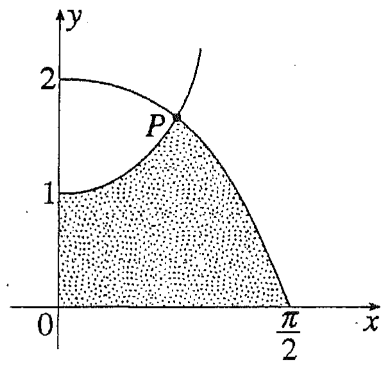
P is a point where the graphs of $y = \tan x$ and $y = \cos x$ intersect.

(i) If the x coordinate of P is α , show that $\sin\alpha = \cos^2\alpha$.

(ii) Prove that, at all such points P, the tangents to the two graphs are perpendicular.

(3)

P is the point of intersection between $x = 0$ and $x = \frac{\pi}{2}$ of the graphs of $y = \sec x$ and $y = 2\cos x$, as shown.



(i) Verify that the x coordinate of P is $\frac{\pi}{4}$.

(ii) The shaded region makes a revolution about the x axis.

Show that the volume of the resulting solid is $\frac{\pi^2}{2}$ cubic units.

(4)

(i) Sketch a graph of $y = |2x - 3|$.

(ii) Show graphically that the equation $|2x-3| = x-2$ has no solutions.

(5)

$f(x) = \sin^3 x + \cos^3 x$

(i) Show that, when $f(x) = 0$, $\tan x = -1$.

(ii) Show that, when $f'(x) = 0$, $\sin x = 0$, $\cos x = 0$ or $\tan x = 1$.

(iii) Sketch a graph of $y = f(x)$ for $0 \le x \le 2\pi$.

(6)

A solid cylinder, radius r and height h, is to be constructed under the condition that the sum of its height and circumference is S, where S is constant.

Prove that

(i) if it is given its maximum possible volume, $h = \frac{S}{3}$

(ii) if it is given its maximum possible total surface area, $r + h = \frac{S}{2}$.

(7)

(i) Show that $\dfrac{a}{ax+1} - \dfrac{1}{x+a} = \dfrac{a^2-1}{(ax+1)(x+a)}$.

(ii) Deduce that, for $a > 1$, $\displaystyle\int_0^1 \frac{a^2-1}{(ax+1)(x+a)}\,dx = \ln a$.

(8)

(i) Show that $\dfrac{x^n + x^{n+2}}{1+x^2} = x^n$.

(ii) An integral is defined by $\mathbf{I}_n = \displaystyle\int_0^1 \frac{x^n}{1+x^2}\,dx$, for $n \geq 0$.

1. Evaluate $\mathbf{I}_0$.

2. Use part (i) to show that $\mathbf{I}_n + \mathbf{I}_{n+2} = \dfrac{1}{n+1}$.

3. **Evaluate $\mathbf{I_2}$.**

(3) Using your knowledge of series and sequences, prove that

$$\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^3+\frac{1}{x^3}\right)^2+\ldots+\left(x^n+\frac{1}{x^n}\right)^2=\frac{\left(x^{2n}-1\right)\left(x^{2n}+1\right)}{x^{2n}\left(x^2-1\right)}$$

Need to write out the solutions.

SOLUTIONS:

(1) (a)

$$\lim_{h\to 0}\frac{\sin\frac{h}{2}}{h} = \lim_{h\to 0}\frac{1}{2}\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}$$
$$= \frac{1}{2}$$

(b)

$A(a, \log a),\ \ B(b, \log b)$

At M, $\quad y = \dfrac{\log a + \log b}{2}$

At G, $\quad y = \log\sqrt{ab}$

$$= \log(ab)^{\frac{1}{2}}$$
$$= \frac{1}{2}(\log ab)$$
$$= \frac{\log a + \log b}{2}$$

Hence GM is parallel to the x axis.

(2)

(i) $\tan\alpha = \cos\alpha$

$$\frac{\sin\alpha}{\cos\alpha} = \cos\alpha$$
$$\sin\alpha = \cos^2\alpha$$

(ii) $\dfrac{d}{dx}(\tan x) = \sec^2 x$

$\dfrac{d}{dx}(\cos x) = -\sin x$

Product of gradients

$= (\sec^2\alpha)(-\sin\alpha)$

$= (\sec^2\alpha)(-\cos^2\alpha)$, from part (i)

$= -1$

(3) (i) $2\cos\dfrac{\pi}{4} = \dfrac{2}{\sqrt{2}}$

$$= \sqrt{2}$$
$$= \sec\frac{\pi}{4}$$

(ii) For the region above $0 < x < \dfrac{\pi}{4}$,

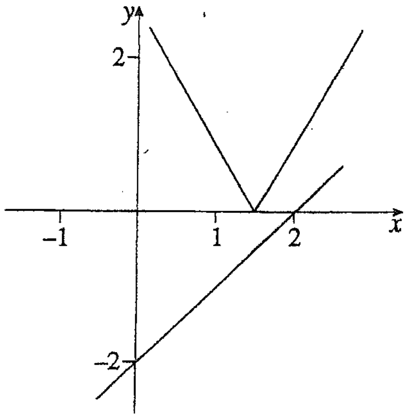
$$V = \pi\int_0^{\frac{\pi}{4}} \sec^2 x\ dx$$
$$= \pi\Big[\tan x\Big]_0^{\frac{\pi}{4}}$$
$= \pi$ cubic units.

For the region above $\dfrac{\pi}{4} < x < \dfrac{\pi}{2}$,

$$V = \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos^2 x\ dx$$
$$= 2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos 2x)\ dx$$
$$= 2\pi\left[x + \frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= 2\pi\left[\frac{\pi}{2} + 0 - \frac{\pi}{4} - \frac{1}{2}\right]$$
$= \left(\dfrac{\pi^2}{2} - \pi\right)$ cubic units.

Hence total volume is $\dfrac{\pi^2}{2}$ cubic units.

(4) (i)

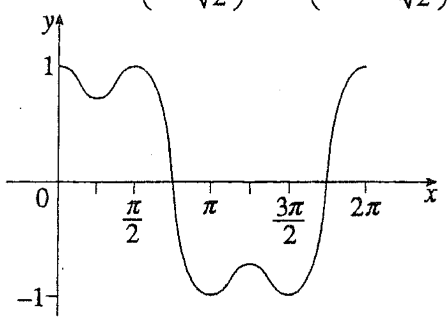


(5)

(i) $\sin^3 x + \cos^3 x = 0$
$\tan^3 x + 1 = 0$
$\tan^3 x = -1$
$\tan x = -1$

(ii) $3\sin^2 x\cos x - 3\cos^2 x\sin x = 0$
$3\sin x\cos x(\sin x - \cos x) = 0$
$\therefore\ \sin x = 0,\ \cos x = 0,$ or $\sin x = \cos x$
$\tan x = 1$

(iii) $\sin x = 0$ at $(0,1)$, $(\pi,-1)$ and $(2\pi,1)$
$\cos x = 0$ at $\left(\frac{\pi}{2},1\right)$ and $\left(\frac{3\pi}{2},-1\right)$
$\tan x = 1$ at $\left(\frac{\pi}{4},\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4},-\frac{1}{\sqrt{2}}\right)$



(6)

$2\pi r + h = S$

(i)
$$V = \pi r^2(S - 2\pi r)$$
$$= \pi\left(Sr^2 - 2\pi r^3\right)$$
$$\frac{dV}{dr} = \pi\left(2Sr - 6\pi r^2\right)$$
$$= 2\pi r(S - 3\pi r)$$
$$\frac{d^2V}{dr^2} = 2\pi(S - 6\pi r)$$

When $r = \frac{S}{3\pi}$, $h = \frac{S}{3}$,

$\frac{dV}{dr} = 0$, $\frac{d^2V}{dr^2} < 0$,

and hence V is maximum.

(ii)
$$A = 2\pi r^2 + 2\pi r(S - 2\pi r)$$
$$= 2\pi\left(r^2 + Sr - 2\pi r^2\right)$$
$$= 2\pi\left(Sr + r^2 - 2\pi r^2\right)$$
$$\frac{dA}{dr} = 2\pi(S + 2r - 4\pi r)$$

When $\frac{dA}{dr} = 0$,
$$S = 4\pi r - 2r$$
$$= r(4\pi - 2)$$
$$r = \frac{S}{4\pi - 2}$$
$$h = S - \frac{2\pi S}{4\pi - 2}$$
$$= \frac{2\pi S - 2S}{4\pi - 2}$$
$$r + h = \frac{2\pi S - S}{4\pi - 2}$$
$$= \frac{S(2\pi - 1)}{2(2\pi - 1)}$$
$$= \frac{S}{2}$$

$\frac{d^2A}{dr^2} = 2\pi(2 - 4\pi) < 0$, hence maximum.

(7)

(i)
$$\frac{a}{(ax+1)} - \frac{1}{x+a} = \frac{a(x+a) - (ax+1)}{(ax+1)(x+a)}$$
$$= \frac{a^2 - 1}{(ax+1)(x+a)}$$

(ii)
$$\int_0^1 \left(\frac{a}{ax+1} - \frac{1}{x+a}\right) dx$$
$$= \left[\ln(ax+1) - \ln(x+a)\right]_0^1$$
$$= \ln(a+1) - \ln(a+1) - \ln 1 + \ln a$$
$$= \ln a$$

(8)

(i) $\dfrac{x^n + x^{n+2}}{1+x^2} = \dfrac{x^n\left(1+x^2\right)}{\left(1+x^2\right)} = x^n$

(ii) (1) $\mathbf{I}_0 = \displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

$= \left[\tan^{-1} x\right]_0^1$

$= \dfrac{\pi}{4}$

(2) $\displaystyle\int_0^1 \frac{x^n}{1+x^2}\,dx + \int_0^1 \frac{x^{n+2}}{1+x^2}\,dx$

$= \displaystyle\int_0^1 \frac{x^n + x^{n+2}}{1+x^2}\,dx$

$= \displaystyle\int_0^1 x^n\,dx$, from part (i),

$= \left[\dfrac{x^{n+1}}{n+1}\right]_0^1$

$= \dfrac{1}{n+1}$

(3) $\mathbf{I}_0 + \mathbf{I}_2 = \dfrac{1}{0+1}$

$\therefore \quad \mathbf{I}_2 = 1 - \dfrac{\pi}{4}$