C.E.M. TUITION

Student Name :________________________________

Review : Mathematical Induction

(PRELIMINARY - PAPER 1)

Year 12 - 3 Unit

Prove the following statements by mathematical induction.

1. $3 \cdot 4 + 5 \cdot 7 + 7 \cdot 10 + \ \ldots \ + (2n+1)(3n+1) = \frac{n}{2}\left(4n^2 + 11n + 9\right)$

2. $\sum_{r=1}^{n} r \cdot 3^r = \frac{3}{4}\left[3^n(2n-1)+1\right]$

3. $7^n + 3n(7^n) - 1$ is divisible by 9

4. (a) $1+\frac{x}{1!}+\frac{1}{2!}x(x+1)+\ \dots\ +\frac{1}{n!}x(x+1)(x+2)\ \dots\ (x+n-1)$

$$=\frac{1}{n!}\ (x+1)(x+2)\ \dots\ (x+n)$$

(b) Deduce that $1-{}^nC_1+{}^nC_2-{}^nC_3+\ \dots\ +(-1)^n\cdot{}^nC_n=0$

Note: This question (4) may be deferred until after the Binomial theorem.

5. $\dfrac{1}{1\cdot 5}+\dfrac{1}{5\cdot 9}+\dfrac{1}{9\cdot 13}+\ \dots\ +\dfrac{1}{(4n-3)(4n+1)}=\dfrac{n}{4n+1}$

1. $S_n = 3\cdot 4 + 5\cdot 7 + \ldots + (2n+1)(3n+1)$

$f(n) = \frac{n}{2}\left(4n^2 + 11n + 9\right)$

$T_n = (2n+1)(3n+1)$

$S_1 = 3\cdot 4 = 12,$

$f(1) = \frac{1}{2}(4+11+9) = 12$

The result is true for $n = 1$.
Assume it is true for $n = K$,

$S_n = f(K) = \frac{K}{2}\left(4K^2 + 11K + 9\right).$

To prove: $S_{K+1} = f(K+1)$, where

$f(K+1) = \frac{1}{2}(K+1)\left[4(K+1)^2 + 11(K+1) + 9\right] = 2K^3 + \frac{23}{2}K^2 + \frac{43}{2}K + 12$

$S_{K+1} = S_K + T_{K+1}$

R.H.S. $= \frac{K}{2}\left(4K^2 + 11K + 9\right) + (2K+3)(3K+4) = 2K^3 + \frac{23}{2}K^2 + \frac{43}{2}K + 12$

$\therefore\ S_{K+1} = f(K+1)$

The result is true for $n = K+1$, if true for $n = K$, etc.

2. $S_n = 1\cdot 3 + 2\cdot 3^2 + \ldots + n\cdot 3^n$

$f(n) = \frac{3}{4}\left[3^n(2n-1)+1\right]$

$T_n = n\cdot 3^n$

$S_1 = 3,$

$f(1) = \frac{3}{4}(3\cdot 1 + 1) = 3$

The result is true for $n = 1$.

Assume $S_k = f(k)$, where

$f(k) = \frac{3}{4}\left[3^k(2k-1)+1\right]$

To prove: $S_{k+1} = f(k+1)$, where

$f(k+1) = \frac{3}{4}\left[3^{k+1}(2k+1)+1\right].$

$S_{k+1} = S_k + T_{k+1} = \frac{3}{4}\left[3^k(2k-1)+1\right] + (k+1)\cdot 3^{k+1}$

$= \frac{3}{4}\left[3^k(2k-1) + 1 + \frac{4}{3}\cdot 3^{k+1}\cdot(k+1)\right]$

$= \frac{3}{4}\left[3^k(2k-1) + 3^k(4k+4) + 1\right]$

$= \frac{3}{4}\left[3^{k+1}\cdot(2k+1)+1\right]$

Hence the result is true for $n = k+1$, etc.

3. $f(n) = 7^n + 3n\left(7^n\right) - 1$

$f(1) = 7 + 21 - 1 = 27$ is divisible by 9.

Assume $f(n)$ is divisible by 9 when $n = k$, a positive integer.

$f(k) = 9M$, M is an integer. Consider

$f(k+1) - f(k)$

$= 7^{k+1} + 3(k+1)\cdot 7^{k+1} - 1 - \left[7^k + 3k\cdot 7^k - 1\right]$

$= \left(7^{k+1} - 7^k\right) + 3\cdot 7^k(21k + 21 - 3k)$

$= 6\cdot 7^k + 7^k(18k+21)$

$= (18k+27)7^k$ Continued

$\therefore\ f(k+1) = f(k) + 9(2k+3)t^k$

But $f(k)$ is divisible by 9 and the 2nd term on R.H.S. is divisible by 9.

Hence $f(k+1)$ is divisible by 9, etc.

4. (a) $S_n = \sum_{r=1}^{n} T_r$, where

$T_n = \frac{x}{n!}(x+1)(x+2)\ldots(x+n-1)$

$f(n) = \frac{1}{n!}(x+1)\ldots(x+n)$

$S_1 = 1 + \frac{x}{1!} = 1 + x$

$f(1) = \frac{x+1}{1!} = 1 + x$

$\therefore$ The result is true for $n = 1$.

Assuming $S_k = f(k)$, we shall prove: $S_{k+1} = f(k+1)$,

where $f(k) = \frac{1}{k!}(x+1)\ldots(x+k)$

$S_{k+1} = S_k + T_{k+1}$

$= \frac{1}{k!}(x+1)\ldots(x+k) + \frac{x}{(k+1)!}(x+1)\ldots(x+k)$

$= \frac{1}{k!}(x+1)\ldots(x+k)\left[1 + \frac{x}{k+1}\right]$

Now $k!\cdot(k+1) = (k+1)!$,

hence $S_{k+1} = \frac{1}{(k+1)!}(x+1)\ldots(x+k+1) = f(k+1)$, etc.

(b) Put $x = -n$ in the given statement, then:

$$1 - \frac{n}{1!} + \frac{(-n)(-n+1)}{2!} + \ldots + \frac{\left((-n)(-n+1)(-n+2)\ldots\ldots(-n+n-1)\right)}{n!} = 0$$

Upon simplification, this becomes

$$1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n\,{}^nC_n = 0$$

5. $S_n = \frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \ldots + \frac{1}{(4n-3)(4n+1)}$

$f(n) = \frac{n}{4n+1},$

$T_n = \frac{1}{(4n-3)(4n+1)}$

$S_1 = \frac{1}{1\cdot 5} = \frac{1}{5},\quad f(1) = \frac{1}{4+1}$

The result is true for $n = 1$.
Assuming $S_k = f(k)$, we shall prove:

$$S_{k+1} = \frac{k+1}{4k+5}$$

$$S_{k+1} = S_k + T_{k+1}$$

$$= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)}$$

$$= \frac{4k^2+5k+1}{(4k+1)(4k+5)}$$

Now $4k^2 + 5k + 1$

$$= (k+1)(4k+1)$$

$$\therefore\ S_{k+1} = \frac{k+1}{4k+5}$$

$$= f(k+1)$$

Hence etc.