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YEAR 12 – EXT. 1 MATHS

REVIEW TOPIC (SP2)

MATHEMATICAL INDUCTION

CEM – Yr 12 – 3U Mathematical Induction – Review Booklet – Paper 2

1. Prove by Mathematical Induction that $7^n - 3^n$ is divisible by 4

2. Prove by Mathematical Induction that $3^n + 5$ is divisible by 8 for all ODD positive integers n.

3. Use mathematical induction to prove that

$$1+6+15+\ldots+n(2n-1)=\frac{1}{6}n(4n-1)(n+1)$$

4. Prove by mathematical induction that $n^3 + 2n$ is divisible by 3 for all positive integers n.

5. Use mathematical induction to prove that $3^{2n-1}+5$ is divisible by 8, for all integers n, $n \geq 1$.

6. Use Mathematical induction to show that $5^n > 3^n + 4^n$ for all positive integers $n \ge 3$.

7. Use the principle of mathematical induction to prove that $3^{2n}-1$ is divisible by 8 when n is a positive integer.

8. Use mathematical induction to prove that $4 \times 2^n + 3^{3n}$ is divisible by 5 for all integers n, $n \geq 0$.

9. Use Mathematical Induction to show that for all positive integers $n - 1$

$$\frac{3}{1\times 2\times 2}+\frac{4}{2\times 3\times 2^2}+\ldots+\frac{n+2}{n(n+1)2^n}=1-\frac{1}{(n+1)2^n}.$$

10. (i) By squaring both sides, show that $2n + 3 > 2\sqrt{(n+1)(n+2)}$ for $n > 0$.

(ii) Prove by mathematical induction that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$$

for all positive integer values of n.

CEM – Yr 12 – 3U Mathematical Induction – Review Booklet – Paper 2

Answers

1. 5

Prove true for $n=1$

$7^1 - 3^1 = 4$ which is divisible by 4 ✓

Assume true for $n=k$

ie $7^k - 3^k = 4Q$ $(Q \in J^+)$ ✓

$7^k = 4Q + 3^k$

Prove true for $n = k+1$

ie $7^{k+1} - 3^{k+1} = 4P$ $(P \in J^+)$

$LHS = 7^k \times 7 - 3^k \times 3$

$= (4Q + 3^k)7 - 3^k \times 3$ ✓

$= 28Q + 4(3^k)$

$= 4(7Q + 3^k)$

$= 4P$ where $P = 7Q + 3^k$

∴ If true for $n=k$ this is true for $n = k+1$. But it is true for $n=1$

∴ true for 2, 3 .. all integral k. ✓

∴ Result is proven true by M I.

2.

(a) $3^n + 5$

Step 1

Prove the statement is true for $n=1$

$3 + 5 = 8$ (True) ✓

Step 2

Assume the statement is true for $n=k$ (where k is an odd integer)

Hence $3^k + 5 = 8M$ (M is an element of Integers)

$3^k = 8M - 5$

Step 3

Prove the statement is true for $n = k+2$

$f(k+2) = 3^{k+2} + 5$

$= 9[3^k] + 5$

$= 9[8M - 5] + 5$ from step 2 ✓

$= 72M - 40$

$= 8[9M - 5]$ (hence divisible by 8 as 8 is a factor) ✓

Hence, if the statement is true for $n=k$, it is true for $n = k+2$

But the statement is true for $n=1$

∴ the statement is true for $n = 3$ etc

Therefore, the statement is true for all odd integers n

3.

(c) ① Prove true $n=1$

$LHS = 1$ $RHS = \frac{1}{6}(1)(3)(2)$

$= 1$

∴ $LHS = RHS$. (1)

② Assume true for $n=k$.

$1 + 6 + 15 + \ldots + k(2k-1) = \frac{1}{6}k(4k-1)($

③ Prove true for $n = k+1$

$1 + 6 + 15 + \ldots + k(2k-1) + (k+1)(2k+1)$

$= \frac{1}{6}(k+1)(4k+3)(k+2$

(1)

$LHS = \frac{1}{6}k(4k-1)(k+1) + (k+1)(2k$

$= \frac{1}{6}(k+1)\left[k(4k-1) + 6(2k+1)\right]$

$= \frac{1}{6}(k+1)(4k^2 - k + 12k + 6)$

$= \frac{1}{6}(k+1)(4k^2 + 11k + 6)$ (1)

$= \frac{1}{6}(k+1)(4k+3)(k+2)$

$= RHS$.

④ If true for $n=k$, then true for $n=k+1$. (1)

Since true for $n=1$, then true for $n=2$ etc.

∴ true for all positive integers

4.

Q6 (a) Prove $n^3 + 2n$ is divisible by 3 for all positive integers n.

Step 1 Prove true for $n=1$

$1^3 + 2 \times 1 = 3$ which is divisible by 3 ∴ True for $n=1$

Step 2 assume true for $n=k$ (= integer)

$k^3 + 2k = 3m$ (m = integer)

Step 3 Prove true for $n = k+1$

$(k+1)^3 + 2(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= (k^3 + 2k) + (3k^2 + 3k + 3)$

$= 3m + 3(k^2 + k + 1)$

which is divisible by 3 since $k^2 + k + 1$ = integer.

∴ True for $n = k+1$

Step 4 Since true for $n=1$ and having assumed true for $n=k$ and subsequently proven true for $n = k+1$, then result is true by Math. Induction for all positive integers n.

(4)

5. Test for $n=1$: $3^{(2\times 1)-1}+5=8$ which is divisible by 8. Thus it is true for $n=1$.

Assume 8 divides $3^{2k-1}+5$.

$\therefore 3^{2k-1}+5=8\lambda$ where λ is an integer

i.e. $3^{2k-1}=8\lambda-5$

Test for $n=k+1$:

$$3^{2(k+1)-1}+5=3^2\times 3^{2k-1}+5$$
$$=9\times(8\lambda-5)+5$$
$$=72\lambda-40$$
$$=8(9\lambda-5)$$

$8(9\lambda-5)$ is divisible by 8 as $9\lambda-5$ is an integer. Thus if $3^{2n-1}+5$ is divisible by 8 for an integer value of n, then it is divisible by 8 for the following integer value of n. Since it is divisible by 8 for $n=1$ and $n=2$, it is divisible by 8 for all positive integers n.

6. Define the sequence of statements $S(n)$: $5^n>3^n+4^n$, $n=3,4,5,\ldots$

Consider $S(3)$: $5^3=125$, $3^3+4^3=91 \Rightarrow 5^3>3^3+4^3$ $\therefore S(3)$ is true.

If $S(k)$ is true: $5^k>3^k+4^k$ **

Consider $S(k+1)$: $5^{k+1}=5.5^k$

$$>5\left(3^k+4^k\right) \quad \textit{if } S(k) \textit{ is true, using ** }$$
$$=5.3^k+5.4^k$$
$$>3.3^k+4.4^k$$
$$=3^{k+1}+4^{k+1}$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(3)$ is true, hence $S(4)$ is true, and then $S(5)$ is true a so on. Hence by Mathematical induction, $5^n>3^n+4^n$ for all positive integers $n\geq 3$.

7. ~~Show true for n=1~~

$3^2-1=8$ $\therefore$ divisible by

let statement be true for $n=k$

$\therefore$ $3^{2k}-1=8M$

Need to show true for $n=k+1$

$$3^{2(k+1)}-1=3^{2k+2}-1$$
$$=3^{2k}.3^2-1$$
$$=3^{2k}.9-1$$
$$=3^{2k}.9-9+8$$
$$=9\left(3^{2k}-1\right)+8$$
$$=9.8M+8$$
$$=8(9M+1)$$

$\therefore$ divisible by 8

Since true for $n=1$ then must be true for $n=1+1=2$ then $n=2+1=3$ $\therefore$ true for all n $n\geq 1$

8. To prove $4\times 2^n+3^{3n}$ is divisible by 5 for integers $n\geq 0$:

- Test for $n=0$.

  $4\times 2^0+3^0=5$

  $\therefore$ true for $n=0$.

- Assume $4\times 2^k+3^{3k}$ is divisible by 5 for intergers $k\geq 0$.

  $\therefore$ $4\times 2^k+3^{3k}=5M$ for an integer M.

- Required to prove $4\times 2^{k+1}+3^{3(k+1)}=5J$, for an integer J.

  Proof

  $$4\times 2^{k+1}+3^{3(k+1)}$$
  $$=4\times 2\times 2^k+3^3\times 3^{3k}$$
  $$=8\times 2^k+3^3\times[5M-4\times 2^k]$$
  $$=2^k[8-27\times 4]+5M\times 3^3$$
  $$=-100\times 2^k+5M\times 3^3$$
  $$=5[-20\times 2^k+M\times 3^3]$$

  As M is an integer, K is an integer, then $[-20\times 2^k+M\times 3^3]$ is an integer also.

  $\therefore$ $4\times 2^{k+1}+3^{3(k+1)}=5J$, for an integer J.

  $\therefore$ The expression is divisible by 5.

- If the expression is divisible by 5 for a value of n, then it is divisible by 5 for the following integral value of n. It has been shown to be divisible by 5 for $n=0$ and so is divisible by 5 for $n=1$, $n=2$ etc.

9. Let $S(n)$, $n=1,2,3,\ldots$ be the sequence of statements $\sum_{r=1}^{n}\frac{r+2}{r(r+1)2^r}=1-\frac{1}{(n+1)2^n}$, $n=1,2,3,\ldots$

Consider $S(1)$: $LHS=\frac{3}{1\times2\times2}=\frac{4-1}{2\times2^1}=1-\frac{1}{2\times2^1}=RHS$. $\therefore S(1)$ is true.

If $S(k)$ is true: $\sum_{r=1}^{k}\frac{r+2}{r(r+1)2^r}=1-\frac{1}{(k+1)2^k}$ $**$

Consider $S(k+1)$:

$$\begin{aligned}LHS&=\sum_{r=1}^{k+1}\frac{r+2}{r(r+1)2^r}\\&=\sum_{r=1}^{k}\frac{r+2}{r(r+1)2^r}+\frac{(k+1)+2}{(k+1)(k+2)2^{k+1}}\\&=1-\frac{1}{(k+1)2^k}+\frac{(k+1)+2}{(k+1)(k+2)2^{k+1}}\qquad \textit{if } S(k) \textit{ is true, using } **\\&=1-\frac{2(k+2)-(k+3)}{(k+1)(k+2)2^{k+1}}\\&=1-\frac{k+1}{(k+1)(k+2)2^{k+1}}\\&=1-\frac{1}{(k+2)2^{k+1}}\\&=RHS\end{aligned}$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Hence by mathematical induction $S(n)$ is true for all positive integers $n\geq1$.

10. (b) (i) $LHS^2 = (2n+3)^2$

$= 4n^2 + 12n + 9$

$RHS^2 = 4(n+1)(n+2)$

$= 4n^2 + 12n + 8$

$\therefore LHS^2 - RHS^2 = 1 > 0$ ✓

$\therefore LHS^2 > RHS^2$

$\therefore LHS > RHS$ (since LHS and RHS are both positive for $n > 0$)

(ii) When $n = 1$, $LHS = \frac{1}{\sqrt{1}} = 1$ ✓

and $RHS = 2(\sqrt{2} - 1) \doteqdot 0.8$

$\therefore LHS > RHS$, so the result is true for $n = 1$.

Suppose that the result is true for the positive integer $n = k$,

i.e. suppose that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{k}} > 2(\sqrt{k+1} - 1)$ (✱)

Prove that the result is true for $n = k+1$,

i.e. prove that $1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > 2(\sqrt{k+2} - 1)$. ✓

$LHS = 1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$> 2(\sqrt{k+1} - 1) + \frac{1}{\sqrt{k+1}}$ (using ✱) ✓

$= \frac{2[(k+1) - \sqrt{k+1}] + 1}{\sqrt{k+1}}$

$= \frac{2k + 3 - 2\sqrt{k+1}}{\sqrt{k+1}}$ ✓

$> \frac{2\sqrt{(k+1)(k+2)} - 2\sqrt{k+1}}{\sqrt{k+1}}$ (using part (i)) ✓

$= 2(\sqrt{k+2} - 1)$

$= RHS$, so the result is true for $n = k+1$ if it is true for $n = k$.