C.E.M.TUITION

Name : ________________________________

Review Topic : Mathematical Induction

(Lesson Notes & Tutorial Exercises)

Maths Extension 1

PHONE : 9666-3331 **FAX : 9316-4996**

PRINCIPLE OF MATHEMATICAL INDUCTION :

There are ***four*** main types to consider, namely :

TYPE 1 : ALGEBRAIC

Prove that the following are true for all positive integers n.

(1) $1+(1+2)+\ldots+(1+2+\ldots+2^{n-1})=2^{n+1}-n-2$

(2) $\sum_{r=1}^{n} r(r!) = (n+1)! - 1$

(3) $\sum_{r=1}^{n}(2^{r-1}+3^{r-1})=(2^n-1)+\frac{1}{2}(3^n-1)$

TYPE 2 : DIVISIBILITY OR MULTIPLICITY :

(1) Prove that $3^n + 5$ is divisible by 8 for all ***odd*** positive integers n.

(2) Prove that $3^n + 7^{n+1}$ is a multiple of 4 for all positive integers n.

TYPE 3 : INEQUALITIES :

This is more likely in the 4 Unit paper. Nevertheless, you should try a couple of these.

(1) Prove by mathematical induction that $5^n \geq 3n + 7$ for $n > 1$.

(2) Prove by mathematical induction that $5^n - 3 > 4^n + 20$ for $n \geq 3$.

TYPE 4 : MISCELLANEOUS PROBLEMS

Again, these questions are more likely to appear in the 4 Unit paper. Nevertheless, try these couple of examples.

(1) A sequence is defined by $u_1 = 2, u_2 = 8$ and for all positive integers n,

$$u_{n+2} = 4(u_{n+1} - u_n)$$

(a) Evaluate u_3 and u_4.

(b) Verify that $u_n = n(2^n)$ for $n = 1, 2, 3, 4$ and investigate whether this result holds for all positive integers.

(c) Prove that the result in (b) is true for all positive integers.

(2) Given that $u_1 = 1, u_2 = 5$ and $u_n = 5u_{n-1} - 6u_{n-2}$ for $n = 2, 3, 4, \ldots$

Prove by induction that $u_n = 3^n - 2^n$ is true for all n.

In most of the solutions, steps 1 and 3 have been left out.

TYPE 1 :

(1) **Step 2 :** Assume true for $n = k$ i.e. $1 + (1+2) + \ldots + (1+2+\ldots+2^{k-1}) = 2^{k+1} - k - 2$

Required to prove that it is also true for $n = k+1$ i.e.

$1 + (1+2) + \ldots + (1+2+\ldots+2^{k}) = 2^{k+2} - (k+1) - 1$

Proof : L.H.S. $= 1 + (1+2) + \ldots + (1+2+\ldots+2^{k-1}) + (1+2+\ldots+2^{k})$

$$= 2^{k+1} - k - 2 + \frac{1(2^{k+1}-1)}{2-1}$$

$$= 2^{k+1} - k - 2 + 2^{k+1} - 1$$

$$= 2.2^{k+1} - k - 3 = 2^{k+2} - (k+1) - 2 = \text{R.H.S as required.}$$

(2) **Step 2 :** Assume true for $n = k$ i.e. $\sum_{r=1}^{k} r(r!) = (k+1)! - 1$

Required to prove that it is also true for $\sum_{r=1}^{k+1} r(r!) = (k+2)! - 1$

Proof : L.H.S. $= \sum_{r=1}^{k} r(r!) + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)!$

$$= (k+1)!(1+k+1) - 1 = (k+1)!(k+2) - 1$$

$$= (k+2)! - 1 = \text{R.H.S.}$$

(3) **Step 2 :** Assume true for $n = k$ i.e. $\sum_{r=1}^{k} (2^{r-1} + 3^{r-1}) = (2^k - 1) + \frac{1}{2}(3^k - 1)$

Required to prove that it is also true for $n = k+1$ i.e.

$$\sum_{r=1}^{k+1} (2^{r-1} + 3^{r-1}) = (2^{k+1} - 1) + \tfrac{1}{2}(3^{k+1} - 1)$$

Proof : L.H.S. $= \sum_{r=1}^{k} (2^{r-1} + 3^{r-1}) + (2^{k+1-1} + 3^{k+1-1})$

$$= (2^k - 1) + \tfrac{1}{2}(3^k - 1) + (2^k + 3^k) = 2.2^k - 1 + \tfrac{1}{2}3^k + 3^k - \tfrac{1}{2}$$

$$= (2^{k+1} - 1) + \frac{3.3^k}{2} - \frac{1}{2} = (2^{k+1} - 1) + \frac{3^{k+1}}{2} - \frac{1}{2}$$

$$= (2^{k+1} - 1) + \tfrac{1}{2}(3^{k+1} - 1) = \text{R.H.S.}$$

TYPE 2 :

(1) **Step 2 :** Assume true for $n = k$ i.e. $3^k + 5 = 8M$ where M is an integer.

Required to prove that it is also true for $n = k + 2$, (n is odd)

i.e. $3^{k+2} + 5 = 8N$ where N is another integer.

Proof : L.H.S. $= 3^k.3^2 + 5 = 9(8M - 5) + 5$

$= 8.9M - 45 - 5 = 8(9M - 5) = 8N =$R.H.S.

(2) **Step 2 :** Assume true for $n = k$ i.e. $3^k + 7^{k+1} = 4M$ where M is an integer.

Required to prove that it is also true for $n = k + 2$

i.e. $3^{k+2} + 7^{k+2} = 4N$ where N is another integer.

Proof : L.H.S. $= 3^k.3^2 + 7^{k+1}.7 = 9.3^k + 7.7^{k+1}$

$= 9(4M - 7^{k+1}) + 7.7^{k+1} = 4.9M - 16.7^{k+1}$

$= 4(9M - 4.7^{k+1}) = 4N$ where N is another integer.

$=$ R.H.S.

TYPE 3 :

(1) To prove that $5^n \geq 3n + 7 \Rightarrow 5^n - 3n - 7 \geq 0$

Step 1 : Let $n = 2$, L.H.S.$= 5^2 - 6 - 7 = 12 > 0$

True for $n = 2$.

Step 2 : Assume true for $n = k$ i.e. $5^k - 3k - 7 \geq 0$

Prove also true for $n = k + 1$ i.e. $5^{k+1} - 3(k + 1) - 7 \geq 0$

L.H.S. $= 5^k.5 - 3k - 3 + 7 = 5(5^k - 3k - 7) + 12k + 39 \geq 0$

(2) To prove that $5^n - 4^n - 23 > 0$ for $n \geq 3$.

<u>**Step 1 :**</u> Prove true for $n = 3$ i.e. $5^3 - 4^3 - 23 = 38 > 0$

<u>**Step 2 :**</u> Assume true for $n = k$ i.e. $5^k - 4^k - 23 > 0$

Prove also true for $n = k+1$ i.e. $5^{k+1} - 4^{k+1} - 23 > 0$

L.H.S. $= 5.5^k - 4.4^k - 23 = 5(5^k - 4^k - 23) + 4^k + 92 > 0$

<u>**TYPE 4 :**</u>

(1) (a) $u_3 = 4(u_2 - u_1) = 4(8-2) = 24$ and $u_4 = 4(u_3 - u_2) = 4(24-8) = 64$

(b) Let $n = 1,\ u_1 = 1(2^1) = 2$ is true.

Let $n = 2,\ u_2 = 2(2^2) = 8$ is true.

Let $n = 3,\ u_3 = 3(2^3) = 24$ is true.

Let $n = 4,\ u_4 = 4(2^4) = 64$ is also true.

(c) Assume true for $n = k$, i.e. $u_k = k(2^k)$.

Prove true for $n = k+1$, i.e. $u_{k+1} = (k+1)(2^{k+1})$

<u>Proof :</u> $u_{k+1} = 4(u_k - u_{k-1}) = 4[k.2^k - (k-1)2^{k-1}]$

$= 4[k.2^k - k.2^{k-1} + 2^{k-1}] = k.2^k.2^2 - k.2^{k-1}.2^2 + 2^{k-1}.2^2$

$= 2k.2^{k+1} - k.2^{k+1} + 2^{k+1} = 2^{k+1}(2k - k + 1)$

$= 2^{k+1}(k+1) =$R.H.S.

(2) <u>**Step 1 :**</u> $u_2 = 3^2 - 2^2 = 5$ which is true.

<u>**Step 2 :**</u> Assume for $n = k$ i.e. $u_k = 3^k - 2^k$ and for $n = k+1$ i.e. $u_{k+1} = 3^{k+1} - 2^{k+1}$

Prove true for $n = k+2$ i.e. $u_{k+2} = 3^{k+2} - 2^{k+2}$

<u>Proof :</u> L.H.S. $= u_{k+2} = 5.u_{k+1} - 6.u_k = 5(3^{k+1} - 2^{k+1}) - 6(3^k - 2^k)$

$= 15.3^k - 6.3^k - 10.2^k + 6.2^k = 9.3^k - 4.2^k$

$= 3^2.3^k - 2^2.2^k = 3^{k+2} - 2^{k+2} =$R.H.S.