C.E.M. TUITION

Student Name : ______________________________

Review Topic : Parametric representation

(Preliminary Course - Paper 1)

Year 11 - 3 Unit

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1. P is the point at $(2ap, ap^2)$ on $x^2 = 4ay$, *prove* that the normal at P has equation $x + py = ap(2+p^2)$. This normal meets the axis of the parabola in G and the latus rectum (produced if necessary) in H. N is the foot of the perpendicular from P on the axis. If S is the focus, show that *SH : *SP = *NP : *NG.

2. PQ is a chord of the parabola $x^2 = 4ay$, whose vertex is O. PQ meets the axis of the parabola in B, and PM, QN are the abscissae from P, Q. Prove that *OM.*ON = *OB^2.

3. (i) The tangent at a point P on $x^2 = 4ay$ (with focus S) meets the latus rectum at A and the directrix at B. Prove that *SA = *SB.

(ii) This tangent also meets the y axis in E, and the normal at P meets the x axis in F. Prove that *PE : *PF = 2 : p.

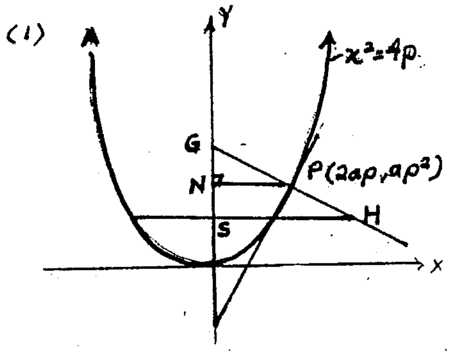
4. $\ulcorner PM \urcorner$, $\ulcorner QN \urcorner$ are perpendiculars from P, Q (on $x^2 = 4ay$) to the directrix. If S is the focus, and {E : E ε $\ulcorner MN \urcorner$, *NE = *EM}, prove that $\ulcorner ES \urcorner$ is perpendicular to $\ulcorner PQ \urcorner$.

5. P is any point on $x^2 = 4ay$; the vertex is O and the focus is S. ⌜PO⌝ meets the directrix in Q, prove that ⌜QS⌝ is parallel to the tangent at P.

6. (i) The tangent at P on $x^2 = 4ay$ meets the tangent at the vertex O in Z. If $\overline{ZM}$ is drawn perpendicular to $\overline{SP}$ meeting it at M, where S is the focus, prove that *OZ = *ZM.

(ii) The normal at P (2,1) on $x^2 = 4y$ meets the y axis in G and M is the midpoint of $\overline{PG}$. A line through M parallel to the x axis meets the y axis in N and the parabola in Q. Prove that *QN = *PG.

(1)



Show normal has eq^n

$$x+py=ap(2+p^2)$$

$$\frac{d}{dx}\left(\frac{x^2}{4a}\right)=\frac{x}{2a}$$

at $x=2ap$, $m_1=p$

$m_2=-\frac{1}{p}$ $(2ap, ap^2)$

$$y-y_1=m(x-x_1)$$
$$y-ap^2=-\frac{1}{p}(x-2ap)$$
$$py-ap^3=-x+2ap$$
$$x+py=ap(2+p^2)$$

Show $\overline{SH}:\overline{SP}=\overline{NP}:\overline{NG}$

$S(0,a)$ $N(0,ap^2)$

$P(2ap, ap^2)$

To find points G, H.

eq^n is $x+py=ap(2+p^2)$

when $x=0$,

$$y=a(2+p^2)$$

$\therefore G[0, a(2+p^2)]$.

when $y=a$

$$x=2ap+ap^3-ap$$
$$x=ap(1+p^2)$$

$\therefore H[ap(1+p^2), a]$

$$\therefore LHS=\frac{\overline{SH}}{\overline{SP}}=\frac{ap(1+p^2)}{\sqrt{4a^2p^2+(ap^2-a)^2}}$$

$$=\frac{ap(1+p^2)}{a\sqrt{(1+p^2)^2}}$$

$$=p$$

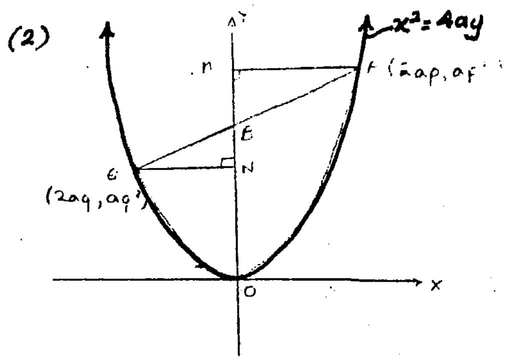
$$RHS=\frac{\overline{NP}}{\overline{NG}}=\frac{2ap}{a(2+p^2)-ap^2}$$

$$=\frac{2ap}{2a}$$

$$=p$$

$\therefore \overline{SH}:\overline{SP}=\overline{NP}:\overline{NG}$

(2)



Prove $\overline{OM}\cdot\overline{ON}=(\overline{OB})^2$

Co-ordinates of $N(0, aq^2)$, $M(0, a$

$O(0,0)$

eq^n of PQ, $m_{PQ}=\frac{ap^2-aq^2}{2ap-2aq}$

$$m_{PQ}=\frac{p+q}{2}$$

$$\therefore y-ap^2=\frac{(p+q)}{2}(x-2ap)$$

$$2y-2ap^2=(p+q)(x-2ap)$$

To find B, $x=0$

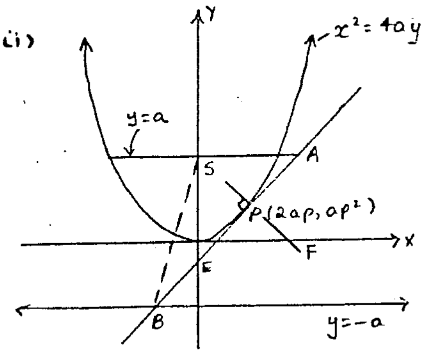
$$\therefore 2y=-2ap^2-2apq+2ap$$
$$y=-apq$$

$B(0,-apq)$

$\therefore LHS=\overline{OM}\cdot\overline{ON}$

$$=ap^2\times aq^2$$
$$=a^2p^2q^2$$

3. (i)



Show $\overline{SA} = \overline{SB}$

Eq^n of tangent $m = p$

$y - ap^2 = p(x - 2ap)$

At A, $y = a$

$\therefore a - ap^2 = px - 2ap^2$

$px = a + ap^2$

$x = \dfrac{a(1+p^2)}{p}$

at B $y = -a$

$-a - ap^2 = p(x - 2ap)$

$px = -a - ap^2 + 2ap^2$

$x = -\dfrac{a(1-p^2)}{p}$

$B\left[\dfrac{-a(1-p^2)}{p}, -a\right]$

$S(0, a)$

$\therefore \overline{SA} = \sqrt{\dfrac{a^2(1+p^2)^2}{p^2} + (a-a)^2}$

$\overline{SA} = \dfrac{a(1+p^2)}{p} \quad \ldots \ldots (\alpha)$

$\overline{SB} = \sqrt{\dfrac{a^2(1-p^2)^2}{p^2} + 4a^2}$

$\overline{SB} = \sqrt{a^2(1-2p^2+p^4) + 4a^2p^2}$

$\overline{SB} = \dfrac{\sqrt{a^2p^4 + 2a^2p^2 + a^2}}{p}$

$\overline{SB} = \dfrac{\sqrt{a^2(1+p^2)^2}}{p}$

$\overline{SB} = \dfrac{a(1+p^2)}{p} \quad \ldots \ldots (\beta)$

$\therefore$ from (α), (β)

$\overline{SA} = \overline{SB}$

(ii) Prove $\overline{PE} : \overline{PF} = 2 : p$

eq^n of tangent $\quad P(2ap, ap^2)$

$y - ap^2 = p(x - 2ap)$

$\therefore$ at $x = 0$

$y = -2ap^2 + ap^2$

$y = -ap^2$

$\therefore E(0, -ap^2)$

eq^n of normal,

$y - ap^2 = -\dfrac{1}{p}(x - 2ap)$

when $y = 0$

$-(x - 2ap) = -ap^3$

$-x + 2ap = -ap^3$

$x = ap(p^2 + 2)$

$\therefore F[ap(p^2+2), 0]$

$\therefore LHS = \dfrac{\overline{PE}}{\overline{PF}} = \dfrac{\sqrt{4a^2p^2 + 4a^2p^4}}{\sqrt{a^2p^6 + a^2p^8}}$

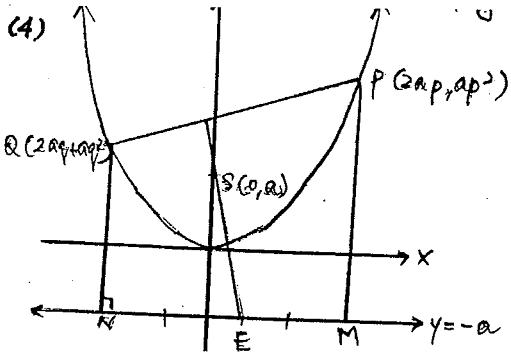
$= \dfrac{2\sqrt{a^2p^2 + a^2p^4}}{p\sqrt{a^2p^2 + a^2p^4}}$

$= 2/p$

LHS = RHS

$\overline{PE} : \overline{PF} = 2 : p$

(4)



Show $\overline{ES}$ is ⊥ to PQ

$N(2aq, -a)$ $M(2ap, -a)$

$\therefore E = \left(\frac{2aq + 2ap}{2}, \frac{-a-a}{2}\right)$

$E(a(q+p), -a)$

$S(0, a)$

$m_{ES} = \frac{-2a}{a(q+p)}$

$m_{ES} = \frac{-2}{p+q}$

$m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{a(p-q)(p+q)}{2a(p-q)}$

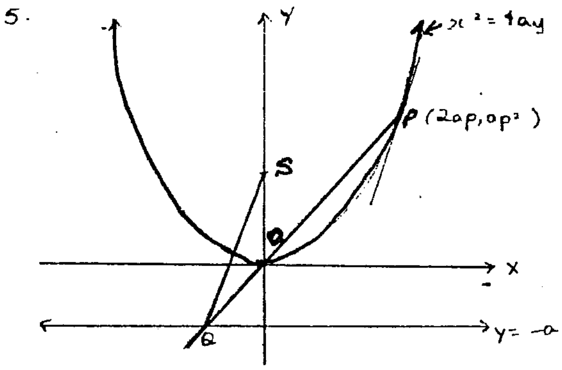
$m_{PQ} = \frac{p+q}{2}$

$\therefore m_{PQ} \times m_{ES} = \frac{p+q}{2} \times \frac{-2}{p+q} = -1$

$\therefore$ ⊥

$\therefore \overline{ES}$ is ⊥ to $\overline{PQ}$.

5.



$\frac{d}{dx}\left(\frac{x^2}{4a}\right) = \frac{x}{2a}$

at $x = 2ap$, $m_P = p$

eq^n $\overline{PO}$, $m = p/2$

$\therefore y - 0 = p/2\,(x - 0)$

$2y = px$

at $y = -a$

$x = -2a/p$

$\therefore Q\left(-\frac{2a}{p}, -a\right)$

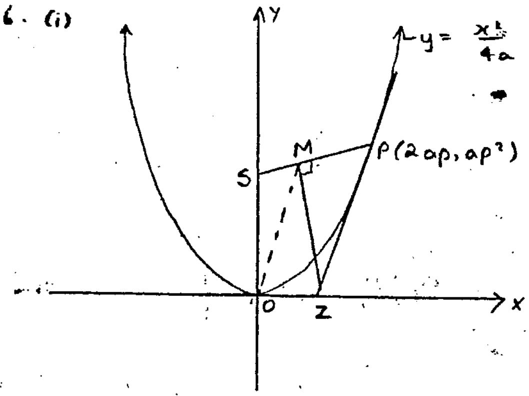
$S(0, a)$

$\therefore m_{QS} = 2a \times \frac{p}{2a} = p$

$\therefore m_{QS} = m_P$

lines are parallel.

6. (i)



Prove $\overline{OZ}=\overline{ZM}$

eq^n of P, $y-ap^2=p(x-2ap)$

at $y=0$

$px-2ap^2=-ap^2$

$x=ap$

$\therefore Z(ap,0)$

eq^n of $\overline{SP}$

$2yp-2ap=(p^2-1)x$

$(p^2-1)x+2ap-2yp=0$

$$\therefore \overline{MZ}=\left|\frac{ap(p^2-1)-0+2ap}{\sqrt{(p^2-1)^2+4p^2}}\right|$$

$$=\left|\frac{ap^3-ap+2ap}{\sqrt{p^4-2p^2+1+4p^2}}\right|$$

$$=\left|\frac{ap(p^2+1)}{\sqrt{(p^2+1)^2}}\right|$$

$$\therefore \overline{MZ}=\frac{ap(p^2+1)}{(p^2+1)}$$

$\overline{MZ}=ap=\overline{OZ}$

$\therefore \overline{MZ}=\overline{OZ}$