NAME :

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# YEAR 12 – EXT. 1 MATHS

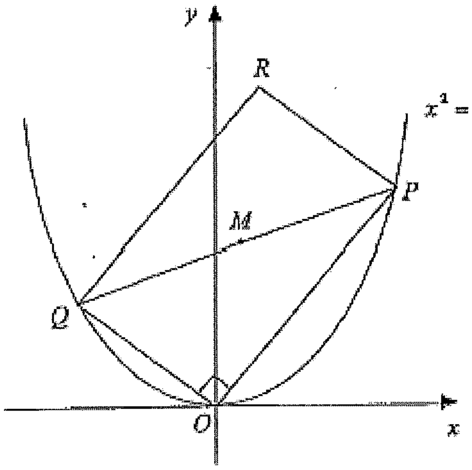
# REVIEW TOPIC (SP3)

# PARAMETRIC REPRESENTATION OF THE PARABOLA

**CEM – Yr 12 – 3U Parametric Representations of the Parabola – Review Paper 3**

1. Show that the equation of the normal to the parabola $x = 2at$, $y = at^2$ at the point where $t = T$ is given by $x + Ty = 2aT + aT^3$.

2.




$P(2ap\,,\,ap^2)$ and $Q(2aq\,,\,aq^2)$ are two points which move on the parabola $x^2 = 4ay$ such that $\angle POQ = 90°$, where $O(0\,,\,0)$ is the origin. $M\left(a(p+q)\,,\,\frac{1}{2}a(p^2+q^2)\right)$ is the midpoint of $PQ$. $R$ is the point such that $OPRQ$ is a rectangle.

(i) Show that $pq = -4$. **1**

(ii) Show that $R$ has coordinates $\left(2a(p+q)\,,\,a(p^2+q^2)\right)$. **1**

(iii) Find the equation of the locus of $R$. **2**

3. The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

(i) Show that the coordinates of the mid-point, $M$, of the chord $PQ$ are
$\left[a(p+q), \frac{a}{2}(p^2+q^2)\right]$.

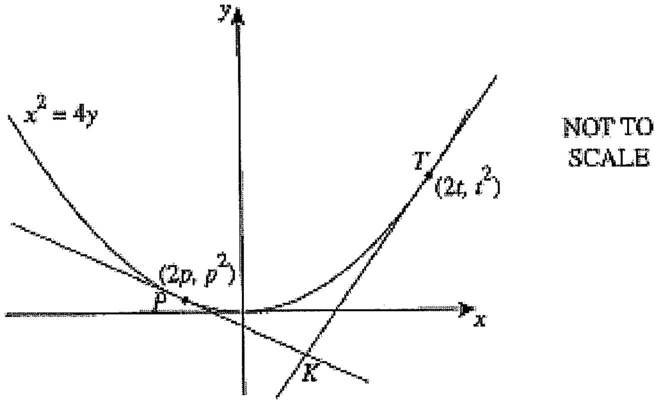
(ii) The chord $PQ$ is a focal chord, i.e. $pq = -1$.
Find the equation of the locus of $M$ and describe the locus of $M$ geometrically.

4. $P(2ap, ap^2)$, $Q(2aq, aq^2)$ and $R(2ar, ar^2)$, where $p < q < r$, are three points on the parabola $x^2 = 4ay$.

(i) Use differentiation to show that the tangent to the parabola at $Q$ has gradient $q$.

(ii) If the chord $PR$ is parallel to the tangent at $Q$, show that $p$, $q$ and $r$ are consecutive terms in an arithmetic sequence.

5.




The diagram shows the graph of the parabola $x^2 = 4y$ and the tangent at $T(2t, t^2)$ and $P(2p, p^2)$. The tangents intersect at point $K$.

(i) Prove that the equation of the tangent at $T$ is $y = tx - t^2$.

(ii) Show that the coordinates of point $K$, the point where the tangents at $T$ and $P$ intersect are $(p + t, pt)$.

(iii) The angle $TKP$ is a right angle.

Show that the locus of $K$ is a straight line.

6. Consider the parabola $x^2 = 4ay$ where $a > 0$.
The tangents at $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ intersect at the point T.
Let $S(o, a)$ be the focus of the parabola.

(i) Find the coordinates of T. (You may assume the equation of the tangent at $P$ is $px - y - ap^2 = 0$ )

(ii) Show that $SP = ap^2 + a$

(iii) Now $P$ and $Q$ move along the parabola in such a way that $SP + SQ = 4a$
Find the locus of $T$ under this condition.

7. $P(6p, 3p^2)$ is a point on the parabola $x^2 = 12y$.

(i) Show that equation of the normal at $P$ is:

$x + py = 6p + 3p^3$

(ii) $Q$ is the point where this normal meets the $y$ – axis.

Find the coordinates of $Q$.

(iii) Show the coordinates of $R$ which divides $PQ$ externally in the ratio 2:1 is $\left(-6p, 3p^2 + 12\right)$.

(iv) Find the Cartesian equation of the locus of $R$.

8. $P(4p, 2p^2)$ and $Q(4q, 2q^2)$ are two points of the parabola $x^2 = 8y$.
The chord $PQ$ subtends a right angle at the origin $O$.

(i) Show that $pq = -4$.

(ii) If $M$ is the midpoint of $PQ$, find the locus of $M$ as $P$ and $Q$ move on the parabola.

9. The two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are on the parabola $x^2 = 4ay$.

i. The equation of the tangent to $x^2 = 4ay$ at a point $T(2at, at^2)$ is given as $y = tx - at^2$.
Show that the tangents at the points $P$ and $Q$ meet at $R$, where $R$ is the point $\left(a(p+q), apq\right)$.

ii. As $P$ varies, the point $Q$ is always chosen so that $\angle POQ$ is a right angle, where $O$ is the origin.
Find the locus of $R$.

10. The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$

(i) Show that the gradient of $PQ$ is $\frac{p+q}{2}$

(ii) Show that if $PQ$ passes through the focus then $pq = -1$

(iii) Find the equation of the locus of the midpoint of $PQ$ if $PQ$ is a focal chord.

Answers

1.

$x = 2at$, $y = at^2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2at}{2a} = t$$ ✓

when $t = T$, $\frac{dy}{dx} = T$, $x = 2aT$, $y = aT^2$

gradient of normal $= -\frac{1}{T}$ ✓

equation of normal:

$$y - aT^2 = -\frac{1}{T}(x - 2aT)$$

$$Ty - aT^3 = -x + 2aT$$

$$x + Ty = 2aT + aT^3$$

2.

i Gradient $OP = \frac{ap^2}{2ap} = \frac{1}{2}p$. Similarly gradient $OQ = \frac{1}{2}q$.

$\therefore OP \perp OQ \Rightarrow \frac{1}{2}p \cdot \frac{1}{2}q = -1$ $\therefore pq = -4$

ii The diagonals of a rectangle bisect each other. Hence $M$ is the midpoint of $OR$.

Hence at $R$, $\frac{1}{2}(x+0) = a(p+q)$ and $\frac{1}{2}(y+0) = \frac{1}{2}a(p^2+q^2)$.

$\therefore x = 2a(p+q)$ and $y = a(p^2+q^2)$

iii At $R$, $y = a\left\{(p+q)^2 - 2pq\right\} = a\left\{\left(\frac{x}{2a}\right)^2 + 8\right\}$

Hence locus of $R$ has equation $x^2 = 4a(y - 8a)$.

3. (i) $P(2ap, ap^2)$ $Q(2aq, aq^2)$

$$M = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$$

$$M = \left(a(p+q), \frac{a}{2}(p^2+q^2)\right)$$

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(ii) $x = a(p+q)$ $y = \left(\frac{a}{2}(p^2+q^2)\right)$

$$y = \frac{a}{2}\{(p+q)^2 - 2pq\}$$

$$y = \frac{a}{2}\{(p+q)^2 + 2\}$$

$$y = \frac{a}{2}\left(\frac{x^2}{a^2} + 2\right)$$

$$y = \frac{x^2}{2a} + a$$

$$x^2 = 2a(y - a)$$

which is a parabola with vertex at $(0, a)$

and focal length $\frac{1}{2}a$.

4. i. $y=\frac{1}{4a}x^2$

$\therefore \frac{dy}{dx}=\frac{1}{2a}x=q$ at $Q$

Hence tangent at $Q$ has gradient $q$.

ii. gradient $PR=\frac{a(r^2-p^2)}{2a(r-p)}=\frac{r+p}{2}$

If $PR$ is parallel to tangent at $Q$

$\frac{r+p}{2}=q$

$r+p=2q$

$r-q=q-p$

$\therefore$ $p$, $q$, $r$ are in arithmetic progression

5. (i) $y=\frac{1}{4}x^2$

$y'=\frac{1}{2}x$

at $(2t, t^2)$, $y'=\frac{1}{2}\times 2t$

$y'=t$

$y-t^2=t(x-2t)$

$y=tx-2t^2+t^2$

$y=tx-t^2$

(ii) solving $y=tx-t^2$ and $y=px-p^2$ simultaneously

$px-p^2=tx-t^2$

$px-tx=p^2-t^2$

$x(p-t)=(p-t)(p+t)$

$p\neq t$

$x=p+t$

$y=t(p+t)-t^2$

$=tp+t^2-t^2$

$=tp$

$K=(p+t, tp)$

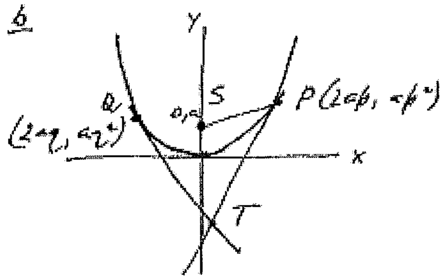
(iii) as $\angle TKP=90°$

$p\times t=-1$

$\therefore K$ is $(p+t, -1)$,

which is a point on the line $y=-1$

6. i)




i $px-y-ap^2=0$ ①

$qx-y-aq^2=0$ ②

① − ②

$(p-q)x=a(p^2-q^2)$

$\therefore x=a(p+q)$

$ap(p+q)-y-ap^2=0$

$\therefore y=apq$

$\therefore T=[a(p+q), apq]$

(2)

6. ii $SP^2=(2ap-0)^2+(ap^2-a)^2$

$=4a^2p^2+a^2p^4-2a^2p^2+a^2$

$=a^2p^4+2a^2p^2+a^2$

$=a^2(p^2+1)^2$

$\therefore SP=ap^2+a$

Condition of locus is

iii $SP+SQ=4a$

$ap^2+a+aq^2+a=4a$

$a(p^2+q^2)=2a$

$\therefore p^2+q^2=2$

$x=a(p+q)$ $\quad$ $y=apq$

$(p+q)^2=p^2+2pq+q^2$

$\frac{x^2}{a^2}=2+\frac{2y}{a}$

$x^2=2a^2+2ay$

$\therefore x^2=2a(y+a)$ is locus of $T$

7. (b) $y = \frac{x^2}{12}$

i) $y' = \frac{x}{6}$ (1)

$m_1 = \frac{6p}{6}$

$m_1 = p$ (1)

$\therefore m_2 = -\frac{1}{p}$

$y - 3p^2 = -\frac{1}{p}(x - 6p)$

$-py + 3p^3 = x - 6p$

$x + py = 6p + 3p^3$ (1)

ii) $x = 0$

$py = 6p + 3p^3$ (1)

$y = 6 + 3p^2$

$Q(0, 6 + 3p^2)$ (1)

iii) $P(6p, 3p^2)$ $Q(0, 6+3p^2)$

$2 : -1$ (1)

$R = \left(\frac{(-1)(6p) + (2)(0)}{2-1}, \frac{(-1)(3p^2) + (2)(6 + 3p^2)}{2-1}\right)$

$= (-6p, -3p^2 + 12 + 6p^2)$

$= (-6p, 3p^2 + 12)$ (1)

iv) $x = -6p$ $y = 3p^2 + 12$

$p = \frac{x}{-6}$

$\therefore y = 3\left(\frac{x^2}{36}\right) + 12$ (1)

$y = \frac{x^2}{12} + 12$

$12y = x^2 + 144$

$x^2 = 12(y - 12)$ (1)

8. $P(4p, 2p^2)$ $Q(4q, 2q^2)$

$m_1$ of $OP$: $\frac{2p^2}{4p} = \frac{p}{2}$

$m_2$ of $OQ$: $\frac{q}{2}$

$m_1 m_2 = -1$

$\frac{p}{2} \times \frac{q}{2} = -1$

$pq = -4$ (1)

9. (a) $P(2ap, ap^2)$ $Q(2aq, aq^2)$ $x^2 = 4ay$

(i) Tangent at P: $y = px - ap^2$ — ①

Tangent at Q: $y = qx - aq^2$ — ②

At R, the pt of intersection

$px - ap^2 = qx - aq^2$

$px - qx = ap^2 - aq^2$

$x(p-q) = a(p^2 - q^2)$ ✓

$x = \frac{a(p^2 - q^2)}{p-q}$ ✓

$x = \frac{a(p-q)(p+q)}{(p-q)}$ ✓

$x = a(p+q)$

$y = p[a(p+q)] - ap^2$

$= ap^2 + apq - ap^2$ ✓

$= apq$

$\therefore R\,[a(p+q), apq]$

(ii) If ∠POQ is a right angle

$\therefore m_{OQ} \times m_{OP} = -1$

$m_{OQ} = \frac{ap^2 - 0}{2ap - 0}$ $m_{OP} = \frac{q}{2}$

$= \frac{ap^2}{2ap}$

$= \frac{p}{2}$

$\therefore \frac{p}{2} \times \frac{q}{2} = -1$ ✓

$\frac{pq}{4} = -1$

$pq = -4$ ✓

Now for R:

$x = a(p+q)$ $y = apq$

but $pq = -4$ $\therefore y = -4a$ ✓ (where a is a constant)

Since $y = -4a$ is always true this must be the locus of R.

10. (b). (i). $m_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{a(p+q)(p-q)}{2a(p-q)}$

$= \frac{p+q}{2}$

(ii).

Equation of line PQ:

$y - ap^2 = \frac{p+q}{2}(x - 2ap)$

If PQ passes through the focus $(0, a)$ then

$a - ap^2 = \frac{p+q}{2}(-2ap)$

$2a - 2ap^2 = -2ap^2 - 2apq$

$2a = -2apq$

$pq = -1$

(iii).

Midpoint of PQ is :

$M = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= \left(a(p+q), \frac{a}{2}(p^2 + q^2)\right)$

$\therefore x = a(p+q)$ i.e. $p + q = \frac{x}{a}$

$y = \frac{a}{2}(p^2 + q^2)$

$= \frac{a}{2}\left((p+q)^2 - 2pq\right)$

$\therefore y = \frac{a}{2}\left(\frac{x^2}{a^2} + 2\right)$ or $y = \frac{x^2}{2a} + a$

since $pq = -1$ (a focal chord)