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YEAR 12 – EXT.1 MATHS

REVIEW TOPIC (SP1) PROJECTILE MOTION (HARDER QUESTIONS)

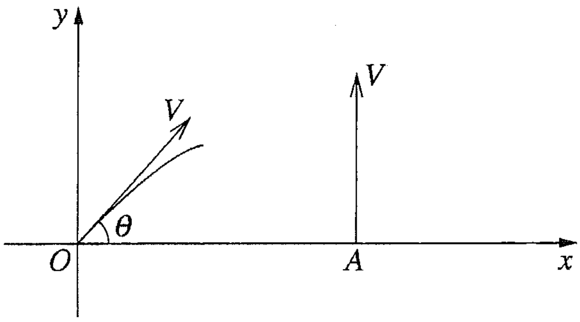
***HSC 06**

(6)

(a) Two particles are fired simultaneously from the ground at time $t = 0$.

Particle 1 is projected from the origin at an angle θ, $0 < \theta < \frac{\pi}{2}$, with an initial velocity V.

Particle 2 is projected vertically upward from the point A, at a distance a to the right of the origin, also with an initial velocity of V.



It can be shown that while both particles are in flight, Particle 1 has equations of motion:

$$x = Vt\cos\theta$$

$$y = Vt\sin\theta - \frac{1}{2}gt^2,$$

and Particle 2 has equations of motion:

$$x = a$$

$$y = Vt - \frac{1}{2}gt^2.$$

Do NOT prove these equations of motion.

Let L be the distance between the particles at time t.

(i) Show that, while both particles are in flight, **2**

$$L^2 = 2V^2t^2(1-\sin\theta) - 2aVt\cos\theta + a^2.$$

(ii) An observer notices that the distance between the particles in flight first decreases, then increases. **3**

Show that the distance between the particles in flight is smallest when

$t = \dfrac{a\cos\theta}{2V(1-\sin\theta)}$ and that this smallest distance is $a\sqrt{\dfrac{1-\sin\theta}{2}}$.

(iii) Show that the smallest distance between the two particles in flight occurs while Particle 1 is ascending if $V > \sqrt{\dfrac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}}$. **1**

***HSC 03**

(7)

(b) A particle is projected from the origin with velocity $v\ \mathrm{m\,s^{-1}}$ at an angle α to the horizontal. The position of the particle at time t seconds is given by the parametric equations

$$x = vt\cos\alpha$$

$$y = vt\sin\alpha - \frac{1}{2}gt^2\ ,$$

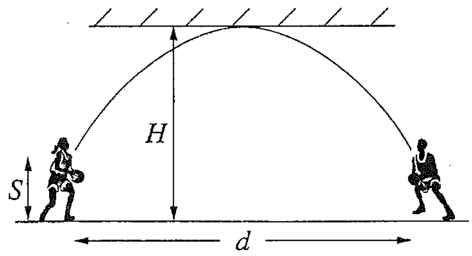
where $g\ \mathrm{m\,s^{-2}}$ is the acceleration due to gravity. (You are NOT required to derive these.)

(i) Show that the maximum height reached, h metres, is given by **2**

$$h = \frac{v^2\sin^2\alpha}{2g}.$$

(ii) Show that it returns to the initial height at $x = \frac{v^2}{g}\sin 2\alpha$. **2**

(iii) Chris and Sandy are tossing a ball to each other in a long hallway. The ceiling height is H metres and the ball is thrown and caught at shoulder height, which is S metres for both Chris and Sandy. **4**



The ball is thrown with a velocity $v\,\mathrm{m\,s^{-1}}$. Show that the maximum separation, d metres, that Chris and Sandy can have and still catch the ball is given by

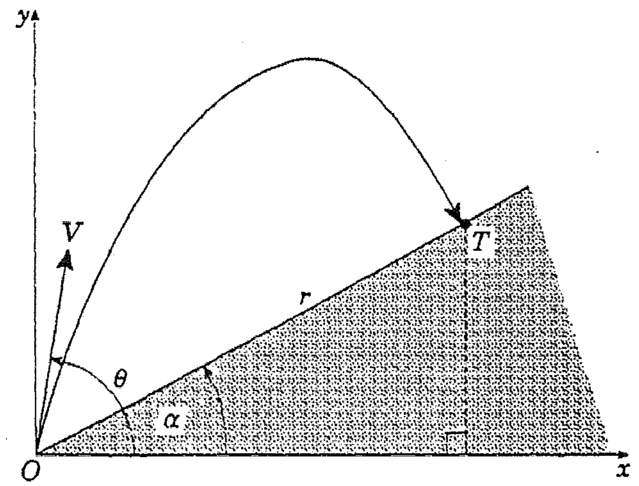
$$d = 4 \times \sqrt{(H-S)\left(\frac{v^2}{2g}\right) - (H-S)^2}, \quad \text{if } v^2 \geq 4g(H-S), \quad \text{and}$$

$$d = \frac{v^2}{g}, \qquad \text{if } v^2 \leq 4g(H-S).$$

***HSC 2000**

(7)

(b) The diagram shows an inclined plane that makes an angle of α radians with the horizontal.



A projectile is fired from O, at the bottom of the incline, with a speed of V m/s at an angle of elevation θ to the horizontal, as shown.

With the above axes, you may assume that the position of the projectile is given by

$$x = Vt\cos\theta \qquad\qquad y = Vt\sin\theta - \frac{1}{2}gt^2.$$

where t is the time, in seconds, after firing, and g is the acceleration due to gravity.

For simplicity we assume that $\dfrac{2V^2}{g} = 1$.

(i) Show that the path of the trajectory of the projectile is $y = x\tan\theta - x^2\sec^2\theta$.

(ii) Show that the range of the projectile, $r = OT$ metres, up the inclined plane is given by

$$r = \frac{\sin(\theta - \alpha)\cos\theta}{\cos^2 \alpha}.$$

(iii) Hence, or otherwise, deduce that the maximum range, R metres, up the incline is

$$R = \frac{1}{2(1+\sin\alpha)}.$$

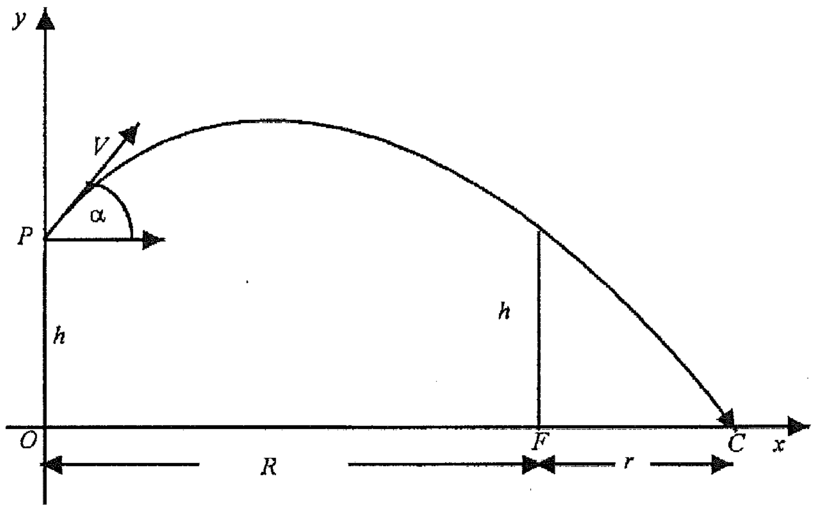
[You may assume that $2\sin A\cos B = \sin(A+B) + \sin(A-B)$.]

(iv) Consider the trajectory of the projectile for which the maximum range R is achieved. Show that, for this trajectory, the initial direction is perpendicular to the direction at which the projectile hits the inclined plane.

(Hint: Firstly show that $m_{\text{at O}} = \tan\theta$, then show that $x = \dfrac{\tan\theta - \tan\alpha}{\sec^2\theta}$ and hence show that $m_{\text{at T}} = -\cot\theta$).

***HSC '95**

(7)



A cap C is lying outside a softball field, r metres from the fence F, which is h metres high. The fence is R metres from the point O, and the point P is h metres from O. Axes are based at O, as shown.

At time $t = 0$, a ball is hit from P at a speed V m/sec at an angle α to the horizontal, towards the cap.

(a) The equations of motion of the ball are **4**

$$\ddot{x} = 0, \qquad \ddot{y} = -g$$

Using calculus, show that the position of the ball at time t is given by

$$x = Vt\cos\alpha$$
$$y = Vt\sin\alpha - \tfrac{1}{2}gt^2 + h$$

(b) Hence show that the trajectory of the ball is given by **1**

$$y = h + x\tan\alpha - x^2\frac{g}{2V^2\cos^2\alpha}.$$

(c) The ball clears the fence. Show that $V^2 \geq \dfrac{gR}{2\sin\alpha\cos\alpha}$ **2**

(d) After clearing the fence, the ball hits the cap C. Show that **3**

$$\tan\alpha \geq \frac{Rh}{(R+r)r}.$$

(e) Suppose that the ball clears the fence, and that $V \le 50, g = 10, R = 80$ and $h = 1$. **2**
What is the closest point to the fence where the ball can land?

0.16 m

***HSC '93**

(7)(b) A projectile is fired from the origin O with velocity V and with angle of elevation θ, where $\theta \neq \frac{\pi}{2}$. You may assume that :

$$x = Vt\cos\theta \qquad \text{and} \qquad y = -\tfrac{1}{2}gt^2 + Vt\sin\theta,$$

where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing.

(i) Show that the equation of flight of the projectile can be written as

$$y = x\tan\theta - \frac{1}{4h}x^2(1+\tan^2\theta), \text{ where } \frac{V^2}{2g} = h.$$

(ii) Show that the point (X,Y), where $X \neq 0$, can be hit by firing at two different angles θ_1 and θ_2 provided $X^2 < 4h(h-Y)$.

(iii) Show that no point **above** the x-axis can be hit by firing at two different angles θ_1 and θ_2, satisfying $\theta_1 < \frac{\pi}{4}$ and $\theta_2 < \frac{\pi}{4}$.

SOLUTIONS

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YEAR 12 – EXT.1 MATHS

REVIEW TOPIC (SP1) PROJECTILE MOTION (HARDER QUESTIONS)

See corrections on pg 7

Awesome!

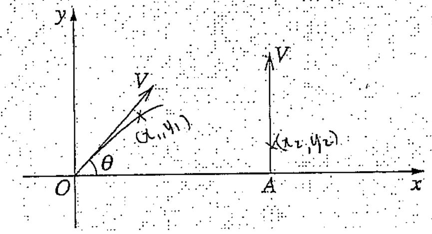
***HSC 06**

(6)

(a) Two particles are fired simultaneously from the ground at time $t = 0$.

Particle 1 is projected from the origin at an angle θ, $0 < \theta < \frac{\pi}{2}$, with an initial velocity V.

Particle 2 is projected vertically upward from the point A, at a distance a to the right of the origin, also with an initial velocity of V.



It can be shown that while both particles are in flight, Particle 1 has equations of motion:

$$x = Vt\cos\theta$$

$$y = Vt\sin\theta - \frac{1}{2}gt^2,$$

and Particle 2 has equations of motion:

$$x = a$$

$$y = Vt - \frac{1}{2}gt^2.$$

Do NOT prove these equations of motion.

Let L be the distance between the particles at time t.

(i) Show that, while both particles are in flight, **2**

$$L^2 = 2V^2t^2(1-\sin\theta) - 2aVt\cos\theta + a^2.$$

$L = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$ ✓

$L^2 = (vt\cos\theta - a)^2 + (vt\sin\theta - \frac{gt^2}{2} - vt + \frac{gt^2}{2})^2$

$= v^2t^2\cos^2\theta - 2avt\cos\theta + a^2 + (vt(\sin\theta - 1))^2$

$= v^2t^2\cos^2\theta + v^2t^2(\sin^2\theta - 2\sin\theta + 1) - 2avt\cos\theta + a^2$

$= v^2t^2(\cos^2\theta + \sin^2\theta - 2\sin\theta + 1) - 2avt\cos\theta + a^2$ ✓

$= v^2t^2(1 - 2\sin\theta + 1) - 2avt\cos\theta + a^2$

$= 2v^2t^2(1-\sin\theta) - 2avt\cos\theta + a^2$ ✓

(ii) An observer notices that the distance between the particles in flight first decreases, then increases. **3**

Show that the distance between the particles in flight is smallest when $t = \dfrac{a\cos\theta}{2V(1-\sin\theta)}$ and that this smallest distance is $a\sqrt{\dfrac{1-\sin\theta}{2}}$.

$L^2 = 2v^2t^2(1-\sin\theta) - 2avt\cos\theta + a^2.$

$\frac{dL}{dv} = \frac{1}{2}\left[2v^2t^2(1-\sin\theta) - 2avt\cos\theta + a^2\right]^{-\frac{1}{2}} \times (4v^2t(1-\sin\theta) - 2av\cos\theta)$ ✓

$\frac{dL}{dv} = 0.$

$2\cancel{4}v^2t(1-\sin\theta) - 2av\cos\theta = 0.$

$2vt(1-\sin\theta) = a\cos\theta.$

$t = \frac{a\cos\theta}{2V(1-\sin\theta)}$ ✓

$L^2 = 2v^2(1-\sin\theta) \times \left[\frac{a\cos\theta}{2V(1-\sin\theta)}\right]^2 - 2aV\cos\theta \times \frac{a\cos\theta}{2V(1-\sin\theta)} + a^2$

$= \frac{a^2\cos^2\theta}{2(1-\sin\theta)} - \frac{a^2\cos^2\theta}{1-\sin\theta} + a^2.$

$= a^2\left(1 - \frac{\cos^2\theta}{2(1-\sin\theta)}\right)$

$= a^2\left[\frac{2(1-\sin\theta) - \cos^2\theta}{2(1-\sin\theta)}\right]$ ✓

$= a^2\left[\frac{2 - 2\sin\theta - (1-\sin^2\theta)}{2(1-\sin\theta)}\right]$

$= a^2\left[\frac{2(1-\sin\theta) - (1+\sin\theta)(1-\sin\theta)}{2(1-\sin\theta)}\right]$

$L^2 = a^2\left[\frac{2-1-\sin\theta}{2}\right].$

$L = a\sqrt{\frac{1-\sin\theta}{2}}$

(iii) Show that the smallest distance between the two particles in flight occurs while Particle 1 is ascending if $V > \sqrt{\dfrac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}}$. **1**

smallest d: $L = a\sqrt{\dfrac{1-\sin\theta}{2}}$

Particle 1 ascending: $\dot{y} > 0$. ✓

$V\sin\theta - gt > 0$.

$V > \dfrac{gt}{\sin\theta}$ ✓

$> \dfrac{g}{\sin\theta} \times \dfrac{a\cos\theta}{2V(1-\sin\theta)}$ ✓

$V^2 > \dfrac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}$

$V > \sqrt{\dfrac{ag\cos\theta}{2\sin\theta(1-\sin\theta)}}$ ✓

***HSC 03**

(7)

(b) A particle is projected from the origin with velocity v ms^{-1} at an angle α to the horizontal. The position of the particle at time t seconds is given by the parametric equations

$$x = vt\cos\alpha$$

$$y = vt\sin\alpha - \frac{1}{2}gt^2,$$

where g ms^{-2} is the acceleration due to gravity. (You are NOT required to derive these.)

(i) Show that the maximum height reached, h metres, is given by **2**

$$h = \frac{v^2\sin^2\alpha}{2g}.$$

$\dot{y} = v\sin\alpha - gt$

max height: $\dot{y} = 0$ $\quad gt = v\sin\theta$. ✓

$t = \dfrac{v\sin\theta}{g}$. ✓

$y = v\left(\dfrac{v\sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\dfrac{v\sin\theta}{g}\right)^2$

$= \dfrac{v^2\sin^2\theta}{g} - \dfrac{v^2\sin^2\theta}{2g}$ ✓

$\therefore \dfrac{v^2\sin^2\theta}{2g}$. ✓

* $\theta = \alpha$

(ii) Show that it returns to the initial height at $x = \frac{v^2}{g}\sin 2\alpha$. **2**

$y = 0$:

$vt\sin\theta - \frac{1}{2}gt^2 = 0$

$t\left(v\sin\theta - \frac{gt}{2}\right) = 0$

$t = 0$, $\frac{gt}{2} = v\sin\theta$

$t = \frac{2v\sin\theta}{g}$

$x = vt\cos\theta$

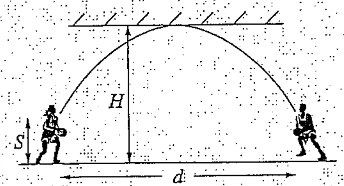
$= v\cos\theta \times \frac{2v\sin\theta}{g}$

$= \frac{v^2}{g} 2\sin\theta\cos\theta$

$= \frac{v^2}{g}\sin 2\theta$

(iii) Chris and Sandy are tossing a ball to each other in a long hallway. The ceiling height is H metres and the ball is thrown and caught at shoulder height, which is S metres for both Chris and Sandy. **4**

$d = \frac{v^2}{g}\sin 2\theta$



The ball is thrown with a velocity $v\,\text{ms}^{-1}$. Show that the maximum separation, d metres, that Chris and Sandy can have and still catch the ball is given by

$$d = 4\times\sqrt{(H-S)\left(\frac{v^2}{2g}\right)-(H-S)^2}, \quad \text{if } v^2 \ge 4g(H-S), \text{ and}$$

$$d = \frac{v^2}{g}, \qquad \text{if } v^2 \le 4g(H-S).$$

$y = \frac{v^2\sin^2\theta}{2g}$ $\qquad x = \frac{v^2}{g}\sin 2\theta$

$H - S = \frac{v^2\sin^2\theta}{2g}$ $\qquad d = \frac{v^2}{g}\sin 2\theta$

$\sin^2\theta = \frac{2g(H-S)}{v^2}$

$\sin\theta = \frac{\sqrt{2g(H-S)}}{v}$ $\qquad$ (if $v^2 \ge 4g(H-S)$)

$\sqrt{2g(H-S)}$, v, θ, $\sqrt{v^2 - 2g(H-S)}$

$v^2 - 2g(H-S) = y^2$

$d = \frac{v^2}{g} \times 2 \times \frac{\sqrt{2g(H-S)}}{v} \times \frac{\sqrt{v^2 - 2g(H-S)}}{v}$

$= \frac{2}{g} \times \sqrt{2v^2g(H-S) - 4g^2(H-S)^2}$

$= 2 \times \sqrt{\frac{2v^2(H-S)}{g} - 4(H-S)^2}$

$= 4 \times \sqrt{\frac{v^2(H-S)}{2g} - (H-S)^2}$

if $v^2 \le 4g(H-S)$

$\sin 2\theta = 2 \times \frac{\sqrt{2g(H-S)}}{v} \times \frac{\sqrt{v^2 - 2g(H-S)}}{v}$

$= \frac{2}{v^2}\sqrt{2gv^2(H-S) - 4g^2(H-S)^2}$

$= \frac{2}{v^2}\sqrt{8g^2(H-S)^2 - 4g^2(H-S)^2}$ $= \frac{4g(H-S)}{v^2}$ and ≤ 1

Without the ceiling d is the max. range i.e. $\theta = \frac{\pi}{4}$

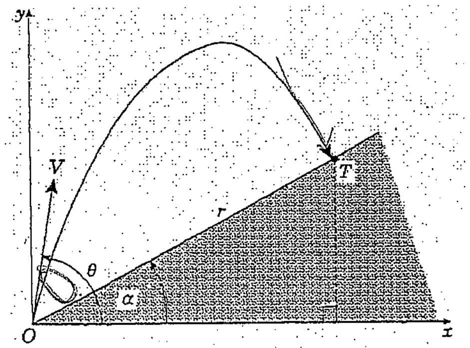
$\therefore d = \frac{v^2}{g}\sin\frac{\pi}{2} = \frac{v^2}{g}$

$d = \frac{v^2}{g}$ (?)

***HSC 2000**

(7)

(b) The diagram shows an inclined plane that makes an angle of α radians with the horizontal.



A projectile is fired from O, at the bottom of the incline, with a speed of V m/s at an angle of elevation θ to the horizontal, as shown.

With the above axes, you may assume that the position of the projectile is given by

$$x = Vt\cos\theta \qquad y = Vt\sin\theta - \frac{1}{2}gt^2.$$

where t is the time, in seconds, after firing, and g is the acceleration due to gravity.

For simplicity we assume that $\frac{2V^2}{g} = 1$.

(i) Show that the path of the trajectory of the projectile is $y = x\tan\theta - x^2\sec^2\theta$.

$t = \frac{x}{V\cos\theta}$

$y = V\sin\theta\left(\frac{x}{V\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2$

$= x\tan\theta - \frac{x^2 g}{2V^2}\sec^2\theta$

$= x\tan\theta - x^2\sec^2\theta$

(ii) Show that the range of the projectile, $r = OT$ metres, up the inclined plane is given by

$$r = \frac{\sin(\theta - \alpha)\cos\theta}{\cos^2\alpha}.$$

$\cos\alpha = \frac{x}{r}$ $\quad x = r\cos\alpha$

$\sin\alpha = \frac{y}{r}$ $\quad y = r\sin\alpha$

$y = x\tan\theta - x^2\sec^2\theta$

$r\sin\theta = r\cos\alpha\tan\theta - r^2\cos^2\alpha\sec^2\theta$

$\sin\alpha = \cos\alpha\tan\theta - r\cos^2\alpha\sec^2\theta$

$r\cos^2\alpha\sec^2\theta = \cos\alpha\tan\theta - \sin\alpha$

$r = \frac{\cos\theta\cos\alpha\sin\theta - \sin\alpha\cos\theta\cos\theta}{\cos^2\alpha}$

$= \frac{\cos\theta}{\cos^2\alpha}(\cos\alpha\sin\theta - \sin\alpha\cos\theta) = \frac{\cos\theta\sin(\theta-\alpha)}{\cos^2\alpha}$

(iii) Hence, or otherwise, deduce that the maximum range, R metres, up the incline is

$$R = \frac{1}{2(1+\sin\alpha)}.$$

[You may assume that $2\sin A\cos B = \sin(A+B) + \sin(A-B)$.]

$r = \frac{\cos\theta\sin(\theta-\alpha)}{(1-\sin^2\alpha)}$

$= \frac{\sin(2\theta-\alpha)+\sin(-\alpha)}{2(1+\sin\alpha)(1-\sin\alpha)}$

$= \frac{\sin(2\theta-\alpha)-\sin\alpha}{2(1+\sin\alpha)(1-\sin\alpha)}$

for MAX R, let $\sin(2\theta-\alpha) = 1$.

$r = \frac{1-\sin\alpha}{2(1+\sin\alpha)(1-\sin\alpha)}$

$= \frac{1}{2(1+\sin\alpha)}$

(iv) Consider the trajectory of the projectile for which the maximum range R is achieved. Show that, for this trajectory, the initial direction is perpendicular to the direction at which the projectile hits the inclined plane.

(Hint: Firstly show that $m_{\text{at O}} = \tan\theta$, then show that $x = \dfrac{\tan\theta - \tan\alpha}{\sec^2\theta}$ and hence show that $m_{\text{at T}} = -\cot\theta$).

$R = \dfrac{1}{2(1+\sin\alpha)}$

$y = x\tan\theta - x^2\sec^2\theta$

$\dfrac{dy}{dx} = \tan\theta - 2x\sec^2\theta$

At $t=0$: $x=0$.

$\therefore m = \tan\theta$

$\dfrac{\sin\alpha - 1 + \cos^2\alpha}{\cos^2\alpha}$

$= \tan\alpha\sec\alpha - \sec^2\alpha + 1$

Eq^n OT: $m = \dfrac{y}{x}$

$= \tan\alpha$

$y = x\tan\alpha$

$x\tan\theta - x^2\sec^2\theta = x\tan\alpha$

$x^2\sec^2\theta + x(\tan\alpha - \tan\theta) = 0$

$x(x\sec^2\theta + \tan\alpha - \tan\theta) = 0$

$x = 0$ or $x = \dfrac{\tan\theta - \tan\alpha}{\sec^2\theta}$

m at T: $x = \dfrac{\tan\theta - \tan\alpha}{\sec^2\theta}$, $y = \dfrac{\tan\alpha(\tan\theta - \tan\alpha)}{\sec^2\theta}$

$\sin(2\theta - \alpha) = 1$

$2\theta - \alpha = \frac{\pi}{2}$

$\alpha = 2\theta - \frac{\pi}{2}$

$m = \tan\theta - 2x\sec^2\theta$

$= \tan\theta - \dfrac{2\sec^2\theta(\tan\theta - \tan\alpha)}{\sec^2\theta}$

$= \tan\theta - 2\tan\theta + 2\tan\alpha$

$= 2\tan\alpha - \tan\theta$

$= 2\tan(2\theta - \frac{\pi}{2}) - \tan\theta$

$= 2\tan[-(\frac{\pi}{2} - 2\theta)] - \tan\theta$

$= -2\cot 2\theta - \tan\theta$

$= \dfrac{-2(1-\tan^2\theta)}{2\tan\theta} - \tan\theta$

$= \dfrac{-(1-\tan^2\theta)}{\tan\theta} - \tan\theta$

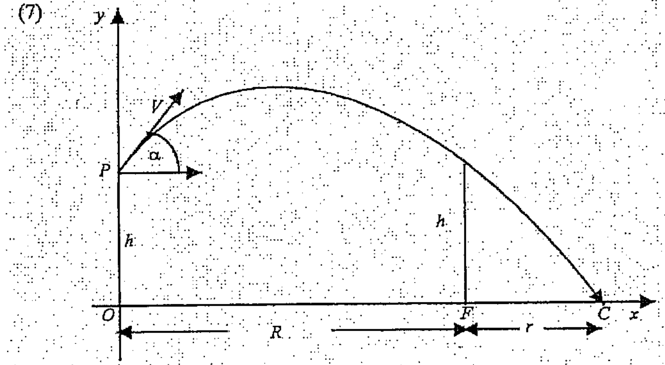
$= \dfrac{-(1-\tan^2\theta) - \tan^2\theta}{\tan\theta}$

$= -\dfrac{1}{\tan\theta}$

$= -\cot\theta$

***HSC '95**

(7)



A cap C is lying outside a softball field, r metres from the fence F, which is h metres high. The fence is R metres from the point O, and the point P is h metres from O. Axes are based at O, as shown.

At time $t = 0$, a ball is hit from P at a speed V m/sec at an angle α to the horizontal, towards the cap.

(a) The equations of motion of the ball are **4**

$\ddot{x} = 0, \quad \ddot{y} = -g$

Using calculus, show that the position of the ball at time t is given by

$x = Vt\cos\alpha$

$y = Vt\sin\alpha - \frac{1}{2}gt^2 + h$

$\dot{x} = V\cos\alpha$

$x = Vt\cos\alpha$

$\dot{y} = V\sin\alpha - gt$

$y = Vt\sin\alpha - \dfrac{gt^2}{2} + C$

$t = 0,\ y = h$

$y = Vt\sin\alpha - \dfrac{gt^2}{2} + h$

(b) Hence show that the trajectory of the ball is given by 1

$$y = h + x\tan\alpha - x^2\frac{g}{2V^2\cos^2\alpha}$$

$t = \frac{x}{V\cos\alpha}$ ✓

$y = V\sin\alpha\left(\frac{x}{V\cos\alpha}\right) - \frac{g}{2}\left(\frac{x}{V\cos\alpha}\right)^2 + h$

$= x\tan\alpha + h - \frac{gx^2}{2V^2\cos^2\alpha}$ ✓

(c) The ball clears the fence. Show that $V^2 \geq \frac{gR}{2\sin\alpha\cos\alpha}$ 2

$x = R$
$y = h$ ✓

$h \leq R\tan\alpha + h - \frac{gR^2}{2V^2\cos^2\alpha}$

$\frac{gR^2}{2V^2\cos^2\alpha} \leq R\tan\alpha$ ✓

$\frac{gR}{2V^2\cos\alpha} \leq \sin\alpha$

$\frac{gR}{2\cos\alpha\sin\alpha} \leq V^2$ ✓

(d) After clearing the fence, the ball hits the cap C. Show that 3

$$\tan\alpha \geq \frac{Rh}{(R+r)r}.$$

$y = h + x\tan\alpha - x^2\frac{g}{2V^2\cos^2\alpha}$ $(R+r, 0)$

$0 = h + \tan\alpha(R+r) - \frac{(R+r)^2 g}{2V^2\cos^2\alpha}$ ✓

$\frac{(R+r)^2 g}{2V^2\cos^2\alpha} = h + \tan\alpha(R+r)$

$V^2 = \frac{(R+r)^2 g}{2\cos^2\alpha} \times \frac{1}{h+\tan\alpha(R+r)}$ ✓

$\frac{(R+r)^2 g}{2\cos^2\alpha} \times \frac{1}{h+\tan\alpha(R+r)} \geq \frac{gR}{2\cos\alpha\sin\alpha}$

$\frac{(R+r)^2}{h\cos\alpha + \sin\alpha(R+r)} \geq \frac{R}{\sin\alpha}$

$\frac{(R+r)^2}{R} \geq \frac{h\cos\alpha + \sin\alpha(R+r)}{\sin\alpha}$

$\geq \frac{h}{\tan\alpha} + (R+r)$

$\frac{(R+r)^2 - R(R+r)}{R} \geq \frac{h}{\tan\alpha}$

$\tan\alpha \geq \frac{hR}{(R+r)(R+r-R)}$ ✓

$\geq \frac{hR}{(R+r)r}$ ✓

(e) Suppose that the ball clears the fence, and that $V \leq 50$, $g = 10$, $R = 80$ and $h = 1$. **2**
What is the closest point to the fence where the ball can land?

$1.\ V^2 \geq \frac{gR}{2\cos\alpha\sin\alpha}$ —

r ?
α ?

y=0.

V ≤ 50.

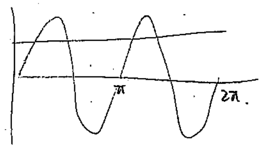
$\tan\alpha \geq \frac{Rh}{(R+r)r}$ ✓

g = 10

R = 80

h = 1

r = ?



$V^2 = \frac{gR}{2\sin\alpha\cos\alpha}$

$2500 = \frac{10 \times 80}{\sin 2\alpha}$ ✓

$\sin 2\alpha = \frac{8}{25}$ ✓

$2\alpha = 18°40'$, $161°20'$ ✓

$\alpha = 9°20'$, $80°40'$

$\tan\alpha \geq \frac{80}{(80+r)r}$

$\alpha > 9°20'$ $(r^2 + 80r)\tan\alpha - 80 = 0$.

$r = \frac{-80\tan\alpha \pm \sqrt{6400\tan^2\alpha + 4(80)}}{2}$ ✓

$= 3.4 \ldots$ ✗.

$\alpha = 80°40'$

r = 0.16 m ✓

0.16 m

***HSC '93**

(7)(b) A projectile is fired from the origin O with velocity V and with angle of elevation θ, where $\theta \neq \frac{\pi}{2}$. You may assume that :

$$x = Vt\cos\theta \quad \text{and} \quad y = -\tfrac{1}{2}gt^2 + Vt\sin\theta,$$

where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing.

(i) Show that the equation of flight of the projectile can be written as

$$y = x\tan\theta - \frac{1}{4h}x^2(1+\tan^2\theta), \text{ where } \frac{V^2}{2g} = h.$$

$t = \frac{x}{V\cos\theta}$

$y = -\frac{g}{2} \times \frac{x^2}{V^2\cos^2\theta} + \frac{V\sin\theta\, x}{V\cos\theta}$ ✓

$= \frac{-x^2 g}{2V^2\cos^2\theta} + x\tan\theta$ ✓ $\quad \frac{V^2}{2g} = h$

$= \frac{-x^2 g}{2 \times 2gh}\sec^2\theta + x\tan\theta$

$= \frac{-x^2}{4h}(\tan^2\theta + 1) + x\tan\theta$ ✓

(ii) Show that the point (X,Y), where $X \neq 0$, can be hit by firing at two different angles θ_1 and θ_2 provided $X^2 < 4h(h-Y)$.

$$\frac{X^2}{4h}(\tan^2\theta + 1) - X\tan\theta + Y = 0$$

at (X,Y) $\quad \tan^2\theta\left(\frac{x^2}{4h}\right) - \tan\theta(x) + \left(y + \frac{x^2}{4h}\right) = 0$

$$\Delta = x^2 - 4\left(\frac{x^2}{4h}\right)\left(\frac{4hy + x^2}{4h}\right)$$

$$= x^2\left(1 - \frac{4hy + x^2}{4h^2}\right)$$

As $x \neq 0$ let $1 - \frac{4hy + x^2}{4h^2} = 0$

$$\frac{4hy + x^2}{4h^2} = 1$$

$$4hy + x^2 = 4h^2$$

$$x^2 = 4h(h-y)$$

~~$x = \pm\sqrt{4h(h-y)}$~~

but since $x^2 < 4h(h-y)$

$\Delta \neq 0$

$\therefore$ 2 distinct roots.

$\therefore$ 2 values for $\tan\theta$

and θ can be

θ_1 & θ_2

(iii) Show that no point **above** the x-axis can be hit by firing at two different angles θ_1 and θ_2, satisfying $\theta_1 < \frac{\pi}{4}$ and $\theta_2 < \frac{\pi}{4}$.

$\frac{v^2}{2g} = h$

$$X^2\tan^2\theta - 4hX\tan\theta + (4hY + X^2) = 0$$

roots: $\tan\theta_1$ and $\tan\theta_2$

product of roots: $\tan\theta_1 \tan\theta_2 = \frac{4hY + X^2}{X^2}$

> 1

if $\theta_1 < \frac{\pi}{4}$ then $\theta_2 > \frac{\pi}{4}$

$\therefore$ no point can be hit if θ_1 and $\theta_2 < \frac{\pi}{4}$