C.E.M. TUITION

Name : ______________________________

Review of Rules and Formulae

Acceleration as a function of x, S.H.M. & Projectile Motion

3 Unit

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2 Unit Review - Application of Calculus to the physical world. Velocity and acceleration w.r.t. time

(i) Displacement :

x refers to displacement ....[1]

(ii) Velocity :

v = Rate of change of x w.r.t time......[2]

$$= \lim_{t \to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t}$$

$$= \frac{dx}{dt} = \dot{x}$$

Note : Particle at rest means v = 0
Positive velocity is motion to the right and negative velocity is motion to the left.

(iii) Acceleration :

Acceleration is the change in velocity over the change in time.

a = Rate of change of v w.r.t. time ...[3]

$$a = \frac{dv}{dt} = \dot{v} = \ddot{x}$$

Note : Positive acceleration is "speeding up" and negative acceleration is "slowing down".

(iv) From Acceleration to Velocity to Displacement :

Acceleration →Velocity→Displacement
Remember to integrate.

3 UNIT
Velocity and acceleration in terms of x

In the 3 Unit component of your course, velocity and acceleration are generally given in terms of the displacement of the particle, x rather than time, t.

(i) Velocity as function of x

When given $v = \frac{dx}{dt} = f(x)$ rewrite it as $\frac{dt}{dx} = \frac{1}{f(x)}$ before integrating w.r.t. x.

Examples

(1) Given that $v = 2x - 3$ and initially the particle starts at $x = 3$ m.
Find an expression for x in terms of t.

$$x = \frac{3(e^{2t} + 1)}{2}$$

(2) Given that $v = \cos^2 2x$ and initially the particle is at $x = \frac{\pi}{8}$ m.
Find an expression for x in terms of t.

$x = \frac{1}{2}\tan^{-1}(2t+1)$

(ii) Acceleration as function of x

Make sure that you know the proof of the alternative form of the acceleration

$$a = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right).$$

Examples

(1) A particle moves on a line so that its distance from the origin at time t is x.

(a) Prove that $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

(b) If $\frac{d^2x}{dt^2} = 10x - 2x^3$ and $v = 0$ at $x = 1$, find v in terms of x.

$v = \pm\sqrt{10x^2 - x^4 - 9}$

(2) An object falling directly to Earth from space moves according to the equation

$$\frac{d^2x}{dt^2} = -\frac{k}{x^2},$$

where x is the distance of the object from the centre of the Earth at time t. The constant k is related to g, the value of gravity at the earth's surface, and the radius of the earth, R, by the formula :

$$k = gR^2.$$

Show that, if the object started from rest at a distance of 10^9 metres from the earth's centre, then it will reach the earth with a velocity of approximately 11 200 ms^{-1}.
(You may assume $R = 6.4 \times 10^6$m, $g = 9.8$ ms^{-2})

(3) (a) The acceleration of a particle moving in a straight line is given by

$$\ddot{x} = -2e^{-x}$$

where x metres is the displacement from the origin. Initially, the particle is at the origin with velocity 2 ms^{-1}.

Prove that $v = 2e^{-\frac{x}{2}}$.

(b) What happens to v as x increases without bound ?

$e^{-\frac{x}{2}} \to 0$ as $x \to \infty$; $v \to 0$

(4) A body moving in a straight line obeys the law :

$\frac{d^2x}{dt^2} = 4x$ where x denotes the position in metres from O, and t in seconds.

(a) Show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d^2x}{dt^2}$

(b) Find the velocity as a function of position if the body is initially $\frac{1}{2}$ m on the positive side of O travelling at 1 ms^{-1} in the positive direction.

$v = 2x$

(c) Find the position as a function of time and calculate the distance travelled by the body during the first $1\frac{1}{2}$ sec. (Leave your answer in exact form.)

$\boxed{\frac{1}{2}(e^3 - 1) \text{ m}}$

(d) Will the body ever come to rest ?
In answering this question, give a brief description of the motion.

$\boxed{\text{As } t > 0, x \to \infty, v \to \infty}$

Simple Harmonic Motion

RULE :

A particle in said to be in S.H.M. if it obeys the rule : $\ddot{x} = -n^2 x$

Remember to use :

Other forms of acceleration : $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

Summary :

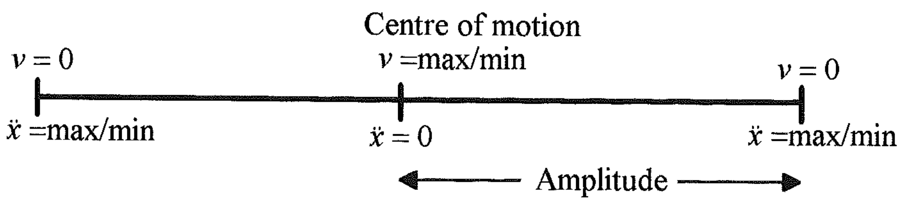
$$\ddot{x} = -n^2 x$$

$$v^2 = n^2(a^2 - x^2), \quad -a \le x \le a$$

$$\begin{aligned} x &= a\sin(nt + \alpha) \\ &= a\sin nt \text{ if } x(0) = 0 \\ &= a\cos nt \text{ if } x(0) = a \end{aligned}$$

$$T = \frac{2\pi}{n} = \frac{1}{f}$$

Be familiar with the nature of the velocity and acceleration of a particle executing S.H.M.



Examples

(1) The velocity v ms^{-1} of a particle moving in simple harmonic motion along the x-axis is given by

$$v^2 = -5 + 6x - x^2 \quad \text{where } x \text{ is in metres.}$$

(a) Between which two points is the particle oscillating ?

$1 \le x \le 5$

(b) Find the centre of motion.

$x = 3$

(c) Find the maximum speed of the particle.

2 ms^{-1}

(d) Find the acceleration of the particle in terms of x.

$\ddot{x} = -(x-3)$

(2) A particle is oscillating in simple harmonic motion such that its displacement x metres from a given origin O satisfies the equation

$$\frac{d^2x}{dt^2} = -4x, \quad \text{where } t \text{ is the time in seconds.}$$

(a) Show that $x = a\cos(2t + \beta)$ is a possible equation of motion for this particle, where a and β are constants.

(b) The particle is observed at time $t = 0$ to have velocity of 2 ms^{-1} and a displacement from the origin of 4 metres. Show that the amplitude of oscillation is $\sqrt{17}$ metres.

(c) Determine the maximum velocity of the particle.

$2\sqrt{17}$ ms^{-1}

(3) A particle P is moving in the x-axis with acceleration $\ddot{x} = -4x$ ms^{-2}.

(a) Find the equation of the motion in the form $x = f(t)$, given that initially the particle is at the origin, moving with a velocity of 6 ms^{-1}.

$x = 3\sin 2t$

(b) Sketch a graph of distance x against time t for the first 2π seconds.

(c) How far will the particle move away from the origin during its motion ?

$-3 \leq x \leq 3$

(d) Calculate the average speed of the particle during the first 2π seconds.

$\frac{12}{\pi}$ ms^{-1}

(4) The displacement of a particle at time t is given by :

$$x = \cos 3t - 2\sin 3t.$$

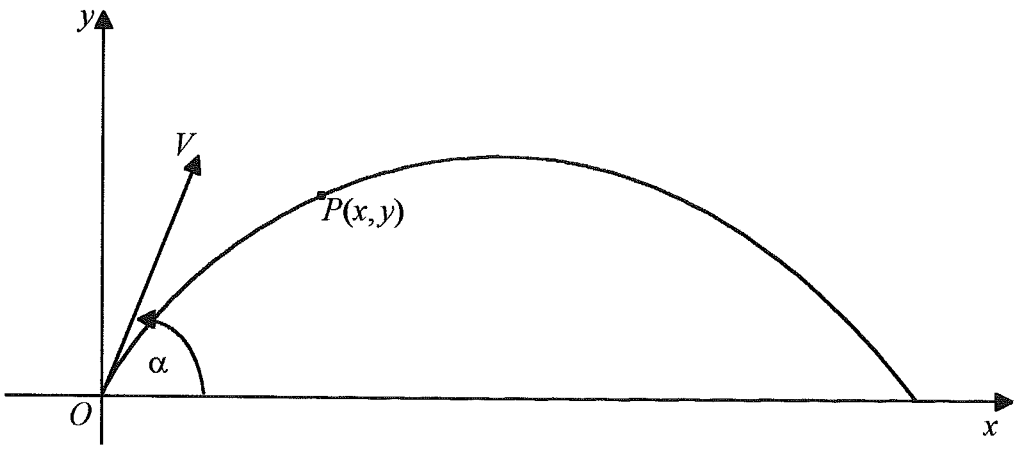
Show that the particle is moving in simple harmonic motion and find the period and amplitude of the motion.

$a = \sqrt{5}, T = \frac{2\pi}{3}$

Projectile Motion

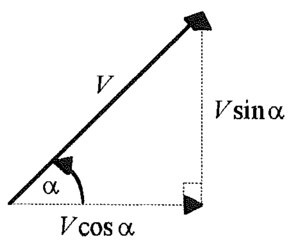
The equations of motion

Must be able to derive the equations of motion of a typical projectile motion as shown below.



Notes : [1] Always choose the point of projection as the origin.

[2] Be familiar with the horizontal and vertical components of the velocity as shown below



Six equations of motion

$\ddot{x} = 0$ .... [i]	$\ddot{y} = -g$ .... [ii]
$\dot{x} = V\cos\alpha$ .... [iii]	$\dot{y} = -gt + V\sin\alpha$ .... [iv]
$x = Vt\cos\alpha$ .... [v]	$y = -\dfrac{gt^2}{2} + Vt\sin\alpha$ .... [vi]

Exercises

HSC '82

(10) A particle P is projected from a point O on a horizontal plane with initial velocity V ms^{-1} in a direction inclined at angle α upwards from the horizontal. At time t seconds after the instant of projection its horizontal and vertical distances from O are x metres and y metres respectively. Air resistance may be neglected. (Draw a diagram)

(i) Write down expressions for x and y as a function of t.

(ii) Show that the time of flight T seconds and the range R metres are given by :

$$T = \frac{2V\sin\alpha}{g},\ R = \frac{V^2\sin 2\alpha}{g},$$

and derive a similar expression for the maximum height reached by P.

$$h = \frac{V^2\sin^2\alpha}{2g}$$

(iii) A ball is thrown from a height 1 metre from the ground and is caught, without bouncing, 2 seconds later 50 m away, also at a height of 1 m. Assuming no air resistance, and that g has the approximate value 10 ms^{-2}, find :

(a) The velocity and angle of projection of the ball;

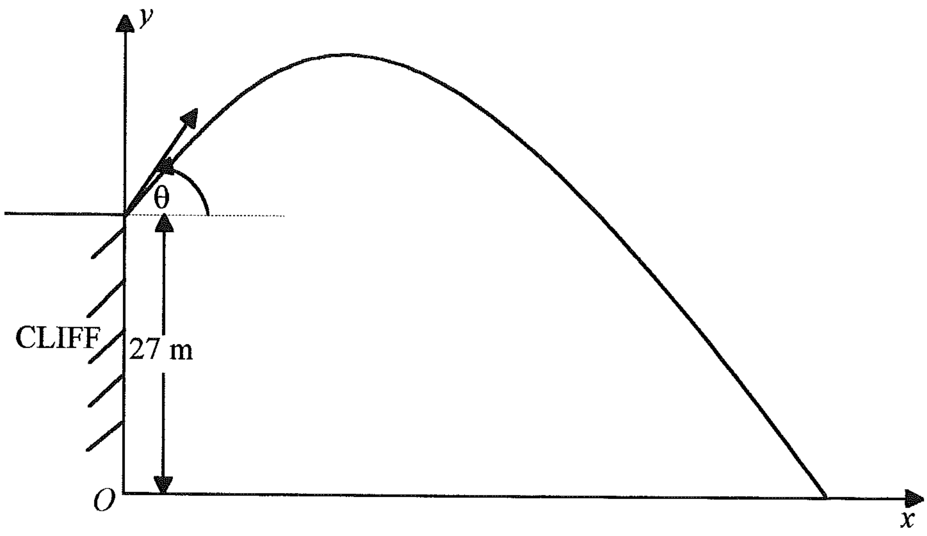
$\alpha = 21^0 48'$; $V = 5\sqrt{29}$ ms^{-1}

(b) the maximum height of the ball above the ground during its flight.

6 metres

HSC '81

A stone is projected with velocity 10 ms^{-1} at an angle of elevation $\theta = \tan^{-1}\frac{3}{4}$ from the top of a vertical cliff 27 m high overlooking a lake.



(i) Write down the equations of motion of the stone, assuming the origin to be a point O at the base of the cliff, and that air resistance may be neglected. Hence derive expressions for the horizontal and vertical components of the stone's displacement from the origin after t seconds.

(ii) Calculate the time which elapses before the stone hits the lake and find the horizontal distance of the point of contact from the base of the cliff. (You may assume the approximate value 10 ms^{-2} for g.)

24 m

(iii) What is the maximum height reached by the stone ?

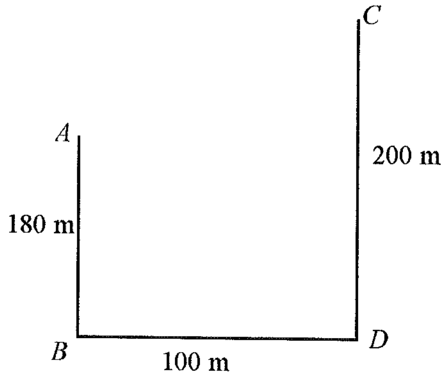
28.8 m

(iv) The path of the stone in the air is a parabolic arc. Write down its equation in cartesian form.

$y = -\frac{5}{64}x^2 + \frac{3}{4}x + 27$

CSSA '85

(6)



AB and *CD* are two buildings situated 100 metres apart on level ground. Their heights are 180 m and 200 m respectively. An object is projected from point *A* at an angle of 45^0 to the horizontal, and this object strikes point *C*. (Take *g* as 10 ms^{-2}).

Show that the time taken for the object to get from *A* to *C* is 4 seconds, and find the value of the initial velocity of projection.

$25\sqrt{2}$ ms^{-1}

CSSA '81

(i) A plane is flying horizontally at 300 ms^{-1} at a height of 2 000 m above a level target area. It releases a bomb directly above a point A on the target area. Calculate :

(a) The distance from A where the bomb lands.

6000 m

(b) The velocity (i.e. magnitude and direction) with which it strikes the ground. (Neglect air resistance. Use $g = 10$ ms^{-2})

361 ms^{-1} at $33^0 41'$

C.E.M.TUITION

Name : ___ , ______________

Review Topics : V(x) & A(x); S.H.M.; Projectile motion

Year 12 - Maths Extension 1

IMPRESSIVE

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2 Unit Review - Application of Calculus to the physical world.
Velocity and acceleration w.r.t. time

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When given $v = \frac{dx}{dt} = f(x)$ rewrite it as $\frac{dt}{dx} = \frac{1}{f(x)}$ before integrating w.r.t. x.

Examples

(1) Given that $v = 2x - 3$ and initially the particle starts at $x = 3$ m.
Find an expression for x in terms of t.

$v = 2x - 3$. $t = 0$ $x = 3$ $v = 3$.
$v = 2(3) - 3$
$= 3$.
$\frac{dx}{dt} = 2x - 3$.
$\frac{dt}{dx} = \frac{1}{2x-3}$
$t = \int \frac{1}{2x-3}\,dx$.
$= \frac{1}{2}\log_e(2x-3) + c$.
$0 = \frac{1}{2}\log_e 3 + c$
$c = -\frac{1}{2}\log_e 3$.

$t = \frac{1}{2}\log_e(2x-3) - \frac{1}{2}\log_e 3$.
$= \frac{1}{2}\log_e\left(\frac{2x-3}{3}\right)$
$2t = \log_e\left(\frac{2x-3}{3}\right)$
$\frac{2x-3}{3} = e^{2t}$.
$2x = 3e^{2t} + 3$
$x = \frac{3(e^{2t}+1)}{2}$.

$x = \frac{3(e^{2t}+1)}{2}$

(2) Given that $v = \cos^2 2x$ and initially the particle is at $x = \frac{\pi}{8}$ m.
Find an expression for x in terms of t.

$\frac{dx}{dt} = \cos^2 2x$. $t=0$ $x = \frac{\pi}{8}$ $v = \frac{1}{2}$
$x = \frac{\pi}{8}$.
$v = \cos^2\frac{\pi}{4}$
$= \frac{1}{2}$.
$\frac{dt}{dx} = \sec^2 2x$.
$t = \int \sec^2 2x\,dx$.
$= \frac{1}{2}\tan 2x + c$.
$0 = \frac{1}{2}\tan\frac{\pi}{4} + c$
$c = -\frac{1}{2}$

$t = \frac{1}{2}\tan 2x - \frac{1}{2}$
$t + \frac{1}{2} = \frac{1}{2}\tan 2x$.
$2t + 1 = \tan 2x$
$2x = \tan^{-1}(2t+1)$
$x = \frac{1}{2}\tan^{-1}(2t+1)$

$x = \frac{1}{2}\tan^{-1}(2t+1)$

(ii) Acceleration as function of x

Make sure that you know the proof of the alternative form of the acceleration

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right).$$

Examples

(1) A particle moves on a line so that its distance from the origin at time t is x.

(a) Prove that $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{2}{2}v\frac{dv}{dx}$
$= v\frac{dv}{dx}$
$= \frac{dx}{dt}\cdot\frac{dv}{dx}$
$= \frac{dv}{dt}$
$= a$
$= \frac{d^2x}{dt^2}$

(b) If $\frac{d^2x}{dt^2} = 10x - 2x^3$ and $v = 0$ at $x = 1$, find v in terms of x.

$a = 10x - 2x^3 = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
$\frac{1}{2}v^2 = \int 10x - 2x^3\,dx$
$= 5x^2 - \frac{x^4}{2} + c$.
$0 = 5 - \frac{1}{2} + c$
$c = -\frac{9}{2}$
$\frac{1}{2}v^2 = 5x^2 - \frac{x^4}{2} - \frac{9}{2}$
$v^2 = 10x^2 - x^4 - 9$
$v = \pm\sqrt{10x^2 - x^4 - 9}$

$v = \pm\sqrt{10x^2 - x^4 - 9}$

(2) An object falling directly to Earth from space moves according to the equation

$$\frac{d^2x}{dt^2} = -\frac{k}{x^2};$$

where x is the distance of the object from the centre of the Earth at time t. The constant k is related to g, the value of gravity at the earth's surface, and the radius of the earth, R, by the formula :

$$k = gR^2.$$

Show that, if the object started from rest at a distance of 10^9 metres from the earth's centre, then it will reach the earth with a velocity of approximately 11 200 ms^{-1}.
(You may assume $R = 6.4 \times 10^6$m, $g = 9.8$ ms^{-2})

$t=0 \quad x = 10^9 \quad v = 0$

$k = gR^2$

$= 9.8 \times (6.4\times10^6)^2$

$= 4.01 \times 10^{14}$ (A)

$a = \frac{-k}{x^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{1}{2}v^2 = -k \int x^{-2}\,dx$

$= \frac{k}{x} + c.$

$0 = \frac{k}{10^9} + c.$ → if :

$c = -401408.$

$\therefore \frac{1}{2}v^2 = \frac{4.01\times10^{-10}}{x} - 401408$

$x = 6.4\times10^6$:

$\frac{1}{2}v^2 = \frac{4.01\times10^{-10}}{6.4\times10^6} - 401408.$

$v = 11164.1$

$\doteqdot 11200$ ms^{-1} (3sf)

(3) (a) The acceleration of a particle moving in a straight line is given by

$$\ddot{x} = -2e^{-x}$$

where x metres is the displacement from the origin. Initially, the particle is at the origin with velocity 2 ms^{-1}.

Prove that $v = 2e^{-\frac{x}{2}}$.

$t=0 \quad x=0 \quad v=2.$

$-2e^{-x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{1}{2}v^2 = -2\int e^{-x}\,dx$

$= -2\left[-e^{-x}\right] + c$

$= 2e^{-x} + c.$

$2 = 2e^{-0} + c$

$2 = 2 + c$

$c = 0.$

$\frac{1}{2}v^2 = 2e^{-x}$

$v^2 = 4e^{-x}.$

$v = 2e^{-\frac{x}{2}}.$

(b) What happens to v as x increases without bound ?

as $x \to \infty$

$e^{-\frac{x}{2}} \to 0$

$v \to 0.$

$e^{-\frac{x}{2}} \to 0$ as $x \to \infty$; $v \to 0$

(4) A body moving in a straight line obeys the law :

$$\frac{d^2x}{dt^2} = 4x \quad \text{where } x \text{ denotes the position in metres from } O, \text{ and } t \text{ in seconds.}$$

(a) Show that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d^2x}{dt^2}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx}$ ✓

$= \frac{dx}{dt} \cdot \frac{dv}{dx}$

$= \frac{dv}{dt}$

$= \frac{d^2x}{dt^2}$ ✓

(b) Find the velocity as a function of position if the body is initially $\frac{1}{2}$ m on the positive side of O travelling at 1 ms^{-1} in the positive direction.

$4x = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{1}{2}v^2 = \int 4x\, dx.$

$= 2x^2 + c.$

$x = \frac{1}{2} \quad v = 1.$ ✓

$\frac{1}{2} = 2\left(\frac{1}{4}\right) + c$

$= \frac{1}{2} + c$

$c = 0.$ ✓

$\frac{1}{2}v^2 = 2x^2$

$v^2 = 4x^2$

$v = \pm 2x.$ ✓

but v in +ve dir ✓

$\therefore v = 2x.$

$v = 2x$

(c) Find the position as a function of time and calculate the distance travelled by the body during the first $1\frac{1}{2}$ sec.(Leave your answer in exact form.)

$\frac{dx}{dt} = 2x$

$\frac{dt}{dx} = \frac{1}{2x}$

$t = \int \frac{1}{2x}\, dx$

$= \frac{1}{2}\log_e x + c.$ ✓

$t = 0 \quad x = \frac{1}{2}.$

$0 = \frac{1}{2}\log_e \frac{1}{2} + c$

$c = -\frac{1}{2}\log_e \frac{1}{2}.$ ✓

$t = \frac{1}{2}\log_e x - \frac{1}{2}\log_e \frac{1}{2}.$

$= \frac{1}{2}\log_e 2x$

$2t = \log_e 2x$ ✓

$2x = e^{2t}$ ✓

$x = \frac{1}{2}e^{2t}.$

$\frac{dx}{dt} = e^{2t}.$

$d = \int_0^{\frac{3}{2}} e^{2t}\, dt.$ ✓

$= \left[\frac{1}{2}e^{2t}\right]_0^{\frac{3}{2}}$

$= \frac{1}{2}\left[e^3 - e^0\right]$ ✓

$= \frac{1}{2}(e^3 - 1)$ ✓ m

$\frac{1}{2}(e^3 - 1)$ m

(d) Will the body ever come to rest ?
In answering this question, give a brief description of the motion.

$v = e^{2t}$

$v \neq 0 \quad \therefore$ will not come ✓ to rest.

as $t \to \infty$, $e^{2t} \to \infty$ ✓ $\therefore v \to \infty$ and

$x = \frac{1}{2}e^{2t}$ ✓ $\quad x \to \infty.$ ✓

As $t > 0, x \to \infty, v \to \infty$

Simple Harmonic Motion

RULE :
A particle in said to be in S.H.M. if it obeys the rule : $\ddot{x} = -n^2 x$

Remember to use :
Other forms of acceleration : $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Summary :

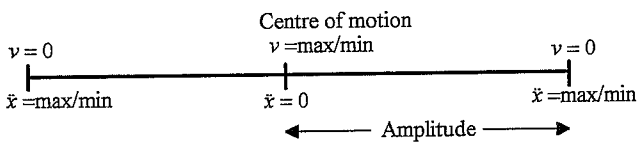
$\ddot{x} = -n^2 x$

$v^2 = n^2(a^2 - x^2), \quad -a \le x \le a$

$x = a\sin(nt + \alpha)$
$\quad = a\sin nt$ if $x(0) = 0$
$\quad = a\cos nt$ if $x(0) = a$

$T = \frac{2\pi}{n} = \frac{1}{f}$

Be familiar with the nature of the velocity and acceleration of a particle executing S.H.M.



Examples

(1) The velocity v ms^{-1} of a particle moving in simple harmonic motion along the x-axis is given by

$$v^2 = -5 + 6x - x^2 \quad \text{where } x \text{ is in metres.}$$

(a) Between which two points is the particle oscillating ?

$v^2 = n^2(a^2 - x^2)$
$v^2 = -5 + 6x - x^2$
$\quad = 4 - (3 - x)^2$

$n = 1$
$a = 2.$
centre of motion : 3.
$\therefore \quad 1 \le x \le 5$

$1 \le x \le 5$

(b) Find the centre of motion.

$x = 3$

$x = 3$

(c) Find the maximum speed of the particle.

max v when $x = 3$:
$v^2 = -5 + 6(3) - 3^2$
$\quad = 4$
$v = 2$ ms^{-1}

2 ms^{-1}

(d) Find the acceleration of the particle in terms of x.

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
$\quad = \frac{d}{dx}\left(\frac{1}{2}(-5 + 6x - x^2)\right)$
$\quad = 3 - x$
$\quad = -(x - 3)$

$\ddot{x} = -(x - 3)$

(2) A particle is oscillating in simple harmonic motion such that its displacement x metres from a given origin O satisfies the equation

$$\frac{d^2x}{dt^2} = -4x,$$ where t is the time in seconds.

(a) Show that $x = a\cos(2t+\beta)$ is a possible equation of motion for this particle, where a and β are constants.

$\dot{x} = -2a\sin(2t+\beta)$

$\ddot{x} = -4a\cos(2t+\beta)$

$= -4x$.

~~∴ asr~~

∴ $x = a\cos(2t+\beta)$ is a ~~soln~~ possible eqn of motion

(b) The particle is observed at time $t = 0$ to have velocity of 2 ms^{-1} and a displacement from the origin of 4 metres. Show that the amplitude of oscillation is $\sqrt{17}$ metres.

$v^2 = n^2(a^2 - x^2)$

$4 = 4(a^2 - 16)$

$a^2 - 16 = 1$

$a^2 = 17$

$a = \sqrt{17}$ m.

(c) Determine the maximum velocity of the particle.

max v when $x = 0$.

$v^2 = 4(17)$

$v = 2\sqrt{17}$ ms^{-1}

$2\sqrt{17}$ ms^{-1}

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(a) Find the equation of the motion in the form $x = f(t)$, given that initially the particle is at the origin, moving with a velocity of 6 ms^{-1}.

$t = 0$ $x = 0$ $v = 6$.

$v^2 = n^2(a^2 - x^2)$

$36 = 4(a^2 - 0)$

$9 = a^2$

$a = 3$.

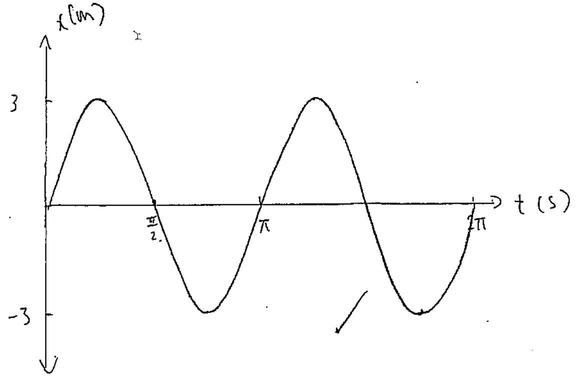
~~$x = 3\sin(t)$~~

$x = a\sin(nt)$

$x = 3\sin(2t)$

$x = 3\sin 2t$

(b) Sketch a graph of distance x against time t for the first 2π seconds.



$T = \frac{2\pi}{n}$

$= \frac{2\pi}{2}$

$= \pi.$

(c) How far will the particle move away from the origin during its motion ?

$-3 \le x \le 3.$

$-3 \le x \le 3$

(d) Calculate the average speed of the particle during the first 2π seconds.

$V = \frac{x}{t}.$

$= \frac{3}{\frac{\pi}{4}}$

$= \frac{12}{\pi}$ ms^{-1}

$\frac{12}{\pi}$ ms^{-1}

(4) The displacement of a particle at time t is given by :

$x = \cos 3t - 2\sin 3t.$

Show that the particle is moving in simple harmonic motion and find the period and amplitude of the motion.

$\dot{x} = -3\sin 3t - 6\cos 3t.$

$\ddot{x} = -9\cos 3t + 18\sin 3t.$

$= -9[\cos 3t - 2\sin 3t].$

$= -9x.$

$\rightarrow$ SHM.

$x = \cos 3t - 2\sin 3t.$

$R = \sqrt{1+4}$

$= \sqrt{5}$

$\tan\alpha = \frac{2}{1}$

$\alpha = \tan^{-1}2.$

$x = \sqrt{5}\cos(3t + \tan^{-1}2).$

$a = \sqrt{5}.$

$T = \frac{2\pi}{n}$

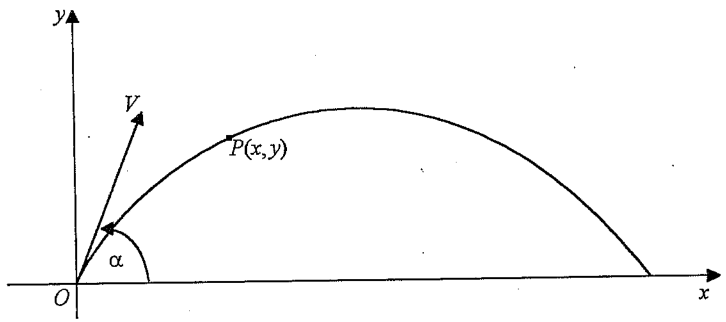
$= \frac{2\pi}{3}$ s.

$a = \sqrt{5}, T = \frac{2\pi}{3}$

Projectile Motion

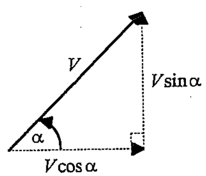
The equations of motion

Must be able to derive the equations of motion of a typical projectile motion as shown below.



Notes : [1] Always choose the point of projection as the origin.

[2] Be familiar with the horizontal and vertical components of the velocity as shown below



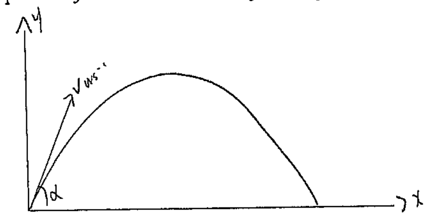
Six equations of motion

$\ddot{x} = 0$ .... [i]	$\ddot{y} = -g$ .... [ii]
$\dot{x} = V\cos\alpha$ .... [iii]	$\dot{y} = -gt + V\sin\alpha$ .... [iv]
$x = Vt\cos\alpha$ .... [v]	$y = -\frac{gt^2}{2} + Vt\sin\alpha$ .... [vi]

Exercises

HSC '82

(10) A particle P is projected from a point O on a horizontal plane with initial velocity V ms^{-1} in a direction inclined at angle α upwards from the horizontal. At time t seconds after the instant of projection its horizontal and vertical distances from O are x metres and y metres respectively. Air resistance may be neglected. (Draw a diagram)



(i) Write down expressions for x and y as a function of t.

$\dot{x} = V\cos\alpha$ $\dot{y} = V\sin\alpha - gt$

$x = Vt\cos\alpha$ ✓ $y = Vt\sin\alpha - \frac{gt^2}{2}$ ✓

(ii) Show that the time of flight T seconds and the range R metres are given by :

$T = \frac{2V\sin\alpha}{g}$, $R = \frac{V^2\sin 2\alpha}{g}$,

and derive a similar expression for the maximum height reached by P.

Time of flight

$y = 0$.

$Vt\sin\alpha - \frac{gt^2}{2} = 0$ ✓

$t\left(V\sin\alpha - \frac{gt}{2}\right) = 0$

$t = 0$ or $\frac{gt}{2} = V\sin\alpha$

$t = \frac{2V\sin\alpha}{g}$ ✓

Range.

$x = Vt\cos\alpha$

$= V\cos\alpha \times \frac{2V\sin\alpha}{g}$

$= \frac{V^2}{g}\sin 2\alpha$ ✓

max h : $t = \frac{V\sin\alpha}{g}$.

$y = V\left(\frac{V\sin\alpha}{g}\right)\sin\alpha - \frac{g}{2}\left(\frac{V^2\sin^2\alpha}{g^2}\right)$

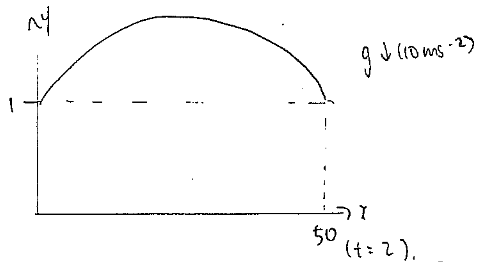
$= \frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g}$ ✓

$= \frac{V^2\sin^2\alpha}{2g}$

$h = \frac{V^2\sin^2\alpha}{2g}$

(iii) A ball is thrown from a height 1 metre from the ground and is caught, without bouncing, 2 seconds later 50 m away, also at a height of 1 m. Assuming no air resistance, and that g has the approximate value 10 ms^{-2}, find :

(a) The velocity and angle of projection of the ball;



$x = Vt\cos\alpha$

$50 = 2V\cos\alpha$

$25 = V\cos\alpha$

$y = Vt\sin\alpha - \frac{gt^2}{2}$

$0 = 2V\sin\alpha - \frac{10(2)^2}{2}$

$10 = V\sin\alpha$

$V^2(\sin^2\alpha + \cos^2\alpha) = 100 + 625$

$V^2 = 725$

$V = 5\sqrt{29}$

$25 = 5\sqrt{29}\cos\alpha$

$\cos\alpha = \frac{5}{\sqrt{29}}$

$\alpha = 21°48'$

$\alpha = 21°48'$; $V = 5\sqrt{29}$ ms^{-1}

(b) the maximum height of the ball above the ground during its flight.

$t = 1$: $y = Vt\sin\alpha - \frac{gt^2}{2} + 1$

$= 5\sqrt{29}\sin 21°48' - 5 + 1$

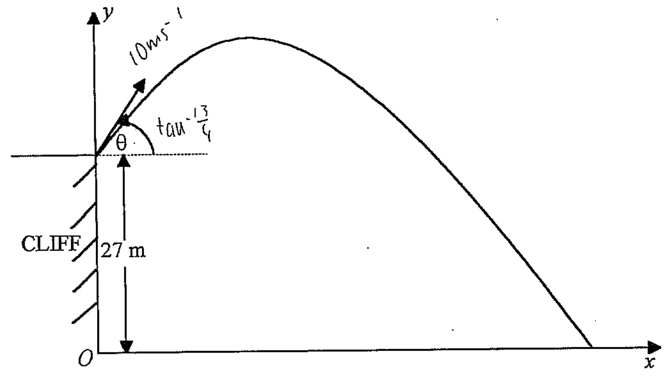
$= 6$

$\therefore$ max height. $y = 6m$

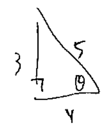
6 metres

HSC '81

A stone is projected with velocity 10 ms^{-1} at an angle of elevation $\theta = \tan^{-1}\frac{3}{4}$ from the top of a vertical cliff 27 m high overlooking a lake.



(i) Write down the equations of motion of the stone, assuming the origin to be a point O at the base of the cliff, and that air resistance may be neglected. Hence derive expressions for the horizontal and vertical components of the stone's displacement from the origin after t seconds.



$\ddot{x} = 0$

$\dot{x} = 10\cos(\tan^{-1}\frac{3}{4})$

$= \frac{10 \times 4}{5}$

$= 8\ ms^{-1}$

$x = 8t$

$\ddot{y} = -10$

$\dot{y} = 10\sin(\tan^{-1}\frac{3}{4}) - 10t$

$= \frac{10 \times 3}{5} - 10t$

$= 6 - 10t$

$y = 6t - 5t^2 + c$

$t = 0$, $y = 27$, $c = 27$

$y = 6t - 5t^2 + 27$

(ii) Calculate the time which elapses before the stone hits the lake and find the horizontal distance of the point of contact from the base of the cliff. (You may assume the approximate value 10 ms^{-2} for g.)

$y = 6t - 5t^2 + 27$

$0 = 6t - 5t^2 + 27$

$5t^2 - 6t - 27 = 0$

$5t^2 - 15t + 9t - 27 = 0$

$5t(t-3) + 9(t-3) = 0$

$(5t+9)(t-3) = 0$

$t = -\frac{9}{5}$ or 3

$\therefore t = 3$

$x = 8t$

$= 8 \times 3$

$= 24$ m

24 m

(iii) What is the maximum height reached by the stone?

$\dot{y} = 0$

$0 = 6 - 10t$

$10t = 6$

$t = \frac{3}{5}$ s

$y = 6\left(\frac{3}{5}\right) - 5\left(\frac{3}{5}\right)^2 + 27$

$= \frac{144}{5}$

$= 28.8$ m

28.8 m

(iv) The path of the stone in the air is a parabolic arc. Write down its equation in cartesian form.

$x = 8t$

$t = \frac{x}{8}$

$y = 6t - 5t^2 + 27$

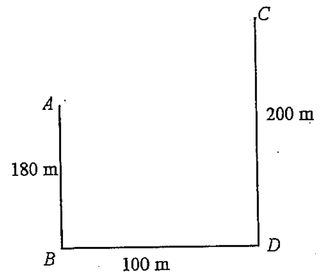
$y = \frac{6x}{8} - \frac{5x^2}{64} + 27$

$= \frac{3x}{4} - \frac{5x^2}{64} + 27$

$y = -\frac{5}{64}x^2 + \frac{3}{4}x + 27$

CSSA '85

(6)



AB and CD are two buildings situated 100 metres apart on level ground. Their heights are 180 m and 200 m respectively. An object is projected from point A at an angle of 45° to the horizontal, and this object strikes point C. (Take g as 10 ms^{-2}).

Show that the time taken for the object to get from A to C is 4 seconds, and find the value of the initial velocity of projection.

$\ddot{x} = 0$

$\dot{x} = V\cos 45$

$= \frac{V}{\sqrt{2}}$

$x = \frac{Vt}{\sqrt{2}}$

$t = \frac{\sqrt{2}x}{V}$

$\ddot{y} = -10$

$\dot{y} = V\sin 45 - 10t$

$= \frac{V}{\sqrt{2}} - 10t$

$y = \frac{Vt}{\sqrt{2}} - 5t^2 + 180$

$y = x - \frac{10x^2}{V^2} + 180$

$x = 100,\ y = 200$:

$200 = 280 - \frac{10(100)^2}{V^2}$

$\frac{100000}{V^2} = 80$

$V^2 = 1250$

$V = 25\sqrt{2}$ ms^{-1}

$t = \frac{\sqrt{2}(100)}{25\sqrt{2}}$

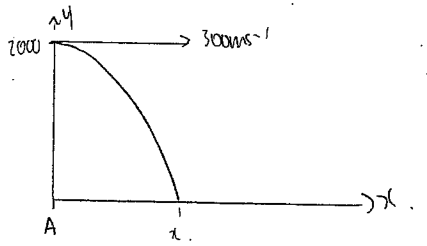
$= 4$ s

$25\sqrt{2}$ ms^{-1}

CSSA '81

(i) A plane is flying horizontally at 300 ms^{-1} at a height of 2 000 m above a level target area. It releases a bomb directly above a point A on the target area. Calculate :

(a) The distance from A where the bomb lands.



$\dot{y} = -10t$

$y = -5t^2 + 2000$

$y = 0$

$2000 = 5t^2$

$t^2 = 400$

$t = 20s$

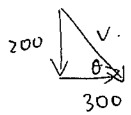
$x = 300t$

$= 300 \times 20$

$= 6000m$

6000 m

(b) The velocity (i.e. magnitude and direction) with which it strikes the ground. (Neglect air resistance. Use $g = 10$ ms^{-2})



$\dot{y} = -10 \times 20$

$= -200$

$V^2 = 300^2 + 200^2$

$V = \sqrt{130000}$

$= 100\sqrt{13}$

$\doteqdot 361$ ms^{-1}

$\tan\theta = \frac{2}{3}$

$\theta = 33°41'$

$\therefore$ V is 361 ms^{-1} @ 33°41' to horizontal

361 ms^{-1} at 33°41'