C.E.M.TUITION

Name : ______________________________

Review of Rules & Formulae

Inverse Functions & Inverse Trigonometric Functions

Year 12 - Maths Extension 1

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[A] Inverse functions

Remember the following properties of $y = f(x)$ and its inverse $y = f^{-1}(x)$

[i] The equation of $y = f^{-1}(x)$ is found by interchanging the $x-$ and $y-$ values of $y = f(x)$.

[ii] The inverse function $f^{-1}(x)$ is a reflection of $f(x)$ about the line $y = x$.

[iii] The domain of $f(x)$ becomes the range of $f^{-1}(x)$ and the range of $f(x)$ becomes the domain of $f^{-1}(x)$.

Examples

[1] [a] Sketch the graph of $f(x) = (x-1)^3$ and its inverse.

[b] Find the equation of its inverse. State the domain and range of this inverse function.

$f^{-1}(x) = x^{\frac{1}{3}} + 1, D$: All real x; R :All real y

[2] [a] Find the greatest domain over which $f(x) = x^2 - 2x$ has an inverse function by sketching $f(x)$ and $f^{-1}(x)$.

$D : x \geq 1$

[b] Find the equation of the inverse function.

$y = \sqrt{x+1} + 1$

[B] Inverse Trigonometric Functions :

Know your three basic graphs i.e. $y = \sin^{-1}x$, $y = \cos^{-1}x$ and $y = \tan^{-1}x$ with their respective domain and range.

Examples

[1] Sketch the graph of $y = -2\sin^{-1}3x$ stating its domain and range.

$D: -\frac{1}{3} \le x \le \frac{1}{3}; R: -\pi \le y \le \pi$

[2] Evaluate the following :

[a] $2\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\boxed{\frac{\pi}{3}}$

[b] $\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right)$

$\boxed{\frac{119}{169}}$

[3] [a] Write down the general solution of $\tan x = \frac{1}{\sqrt{3}}$.

$$x = n\pi + \frac{\pi}{6}$$

[b] Find the solution given by $n = -2$.

$$-\frac{11\pi}{6}$$

[4] [a] Find the second derivative of $\sin^{-1}\left(\frac{x}{2}\right)$.

$$\boxed{\frac{x}{\sqrt{(4-x^2)^3}}}$$

[b] Hence, find the equation of the tangent to the curve $y = \sin^{-1}\left(\frac{x}{2}\right)$ at the point $\left(1, \frac{\pi}{6}\right)$.

$$\boxed{6y - 2\sqrt{3}\,x + 2\sqrt{3} - \pi = 0}$$

[5] Find $\frac{d}{dx}\left(x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)\right)$. Hence evaluate $\int_{\frac{1}{\sqrt{3}}}^{1}\tan^{-1}x\,dx$ correct to 2 decimal places.

$$\frac{\pi}{4}-\frac{\sqrt{3}\,\pi}{18}+\frac{1}{2}\ln\frac{2}{3}$$

[6] Find :

[a] $\int_{\frac{3}{8}}^{\frac{3}{4}} \frac{1}{\sqrt{9-16x^2}}\, dx$

$\boxed{\frac{\pi}{12}}$

[b] $\int_{\frac{1}{5}}^{\frac{\sqrt{3}}{5}} \frac{1}{1+25x^2}\, dx$

$\boxed{\frac{\pi}{60}}$

[7] Differentiate with respect to x :

[a] $\ln\left(\sin^{-1}2x\right)$

$$\boxed{\frac{2}{\sin^{-1}2x\sqrt{1-4x^2}}}$$

[b] $e^{\tan^{-1}\frac{x}{2}}$

[8] [a] Show that if $0 < \alpha < 1$ and $0 < \beta < 1$, then

$$\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

[b] A flag-post 9 metres high stands on the top of a support, the top of which is 16 metres above the eye-level of an observer. If the flag-pole subtends an angle of θ at the eye of the observer who is standing x metres from the support, prove (with the aid of a diagram) that :

$$\theta = \tan^{-1}\left(\frac{9x}{x^2+400}\right).$$

[c] Hence, or otherwise, determine the distance the observer must stand from the support to give θ a maximum value.

$x = 20$ m

[A] Inverse functions

Remember the following properties of $y = f(x)$ and its inverse $y = f^{-1}(x)$

[i] The equation of $y = f^{-1}(x)$ is found by interchanging the x– and y– values of $y = f(x)$.

[ii] The inverse function $f^{-1}(x)$ is a reflection of $f(x)$ about the line $y = x$.

[iii] The domain of $f(x)$ becomes the range of $f^{-1}(x)$ and the range of $f(x)$ becomes the domain of $f^{-1}(x)$.

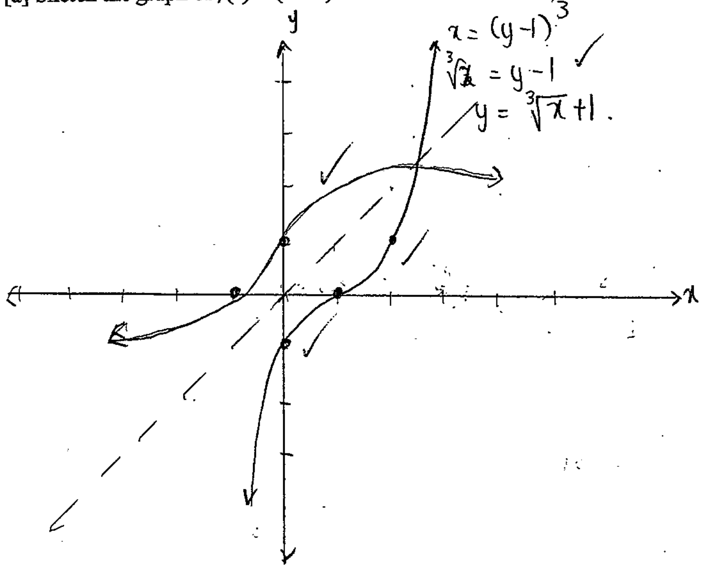
Examples

[1] [a] Sketch the graph of $f(x) = (x-1)^3$ and its inverse.

$x = (y-1)^3$

$\sqrt[3]{x} = y - 1$

$y = \sqrt[3]{x} + 1$.



[b] Find the equation of its inverse. State the domain and range of this inverse function.

$y = \sqrt[3]{x} + 1$

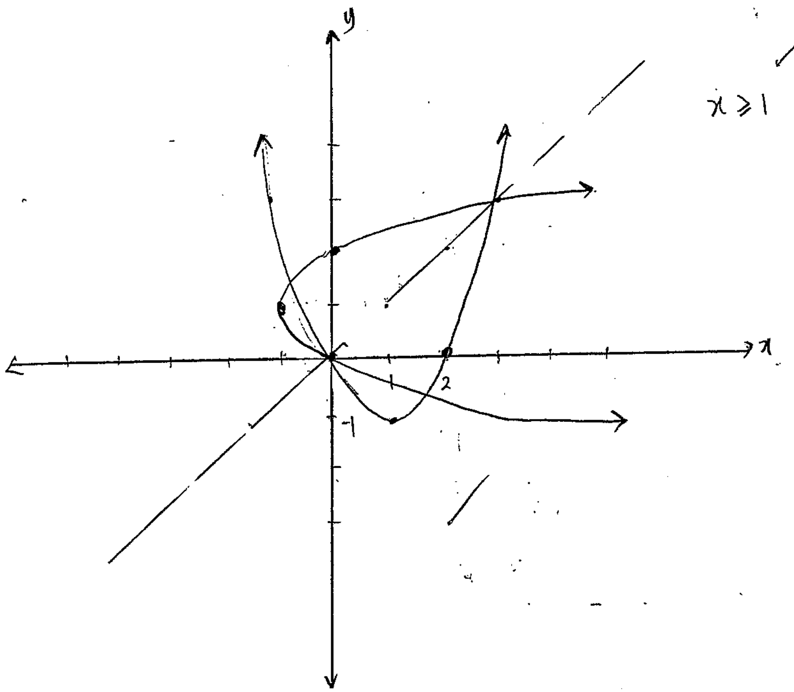
$x \in R$

$y \in R$

$f^{-1}(x) = x^{\frac{1}{3}} + 1, D:$ All real x; $R:$ All real y

[2] [a] Find the greatest domain over which $f(x) = x^2 - 2x$ has an inverse function by sketching $f(x)$ and $f^{-1}(x)$.

$f(x) = x(x-2)$



$x \geq 1$ or $x \leq -1$

$D: x \geq 1$

[b] Find the equation of the inverse function.

$1 + \quad x = y^2 - 2y + 1$

$x + 1 = (y-1)^2$

$y - 1 = \pm\sqrt{x+1}$

$y = 1 \pm \sqrt{x+1}$

sub $(0, 2)$

$y = 1 + \sqrt{x+1}$

$2 = 1 + \sqrt{1}$

$= 2$

$\therefore y = 1 + \sqrt{x+1}$

$\therefore f^{-1}(x) = 1 + \sqrt{x+1}$

$y = \sqrt{x+1} + 1$

[B] Inverse Trigonometric Functions :

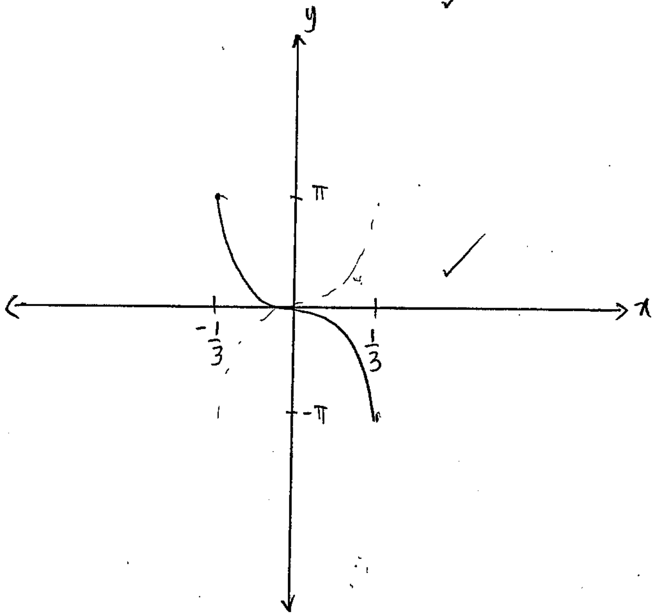
Know your three basic graphs i.e. $y=\sin^{-1}x$, $y=\cos^{-1}x$ and $y=\tan^{-1}x$ with their respective domain and range.

Examples

[1] Sketch the graph of $y=-2\sin^{-1}3x$ stating its domain and range.

$-1 \leq 3x \leq 1$ $\quad -\frac{\pi}{2} \leq \frac{y}{2} \leq \frac{\pi}{2}$

$x: -\frac{1}{3} \leq x \leq \frac{1}{3}$ ✓ $\quad y: -\pi \leq y \leq \pi$ ✓



$D: -\frac{1}{3} \leq x \leq \frac{1}{3}; R: -\pi \leq y \leq \pi$

[2] Evaluate the following :

[a] $2\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= 2 \times -\frac{\pi}{4} + \frac{5\pi}{6}$ ✓ ✓

$= \frac{\pi}{3}$ ✓

$\frac{\pi}{3}$

[b] $\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right)$

let $\beta = \sin^{-1}\frac{5}{13}$

$\sin\beta = \frac{5}{13}$ ✓

$\cos 2\beta = 2\cos^2\beta - 1$

$= 2\left(\frac{12}{13}\right)^2 - 1$

$= \frac{119}{169}$ ✓

$\frac{119}{169}$

[3] [a] Write down the general solution of $\tan x = \frac{1}{\sqrt{3}}$.

$x = \frac{\pi}{6}$

$\therefore x = \frac{\pi}{6} + \pi n, \quad n \in Z.$

$x = n\pi + \frac{\pi}{6}$

[b] Find the solution given by $n = -2$.

$n = -2, \quad x = \frac{\pi}{6} - 2\pi$

$= -1\frac{5}{6}\pi$

$-\frac{11\pi}{6}$

[4] [a] Find the second derivative of $\sin^{-1}\left(\frac{x}{2}\right)$.

$y' = \frac{1}{\sqrt{1 - \frac{x^2}{4}}} \times \frac{1}{2}$

$= \frac{1}{\sqrt{4 - x^2}}$

$= (4 - x^2)^{-\frac{1}{2}}$

$y'' = -\frac{1}{2}(4 - x^2)^{-\frac{3}{2}} \cdot -2x$

$= \frac{x}{(4 - x^2)^{\frac{3}{2}}}$

$\frac{x}{\sqrt{(4-x^2)^3}}$

[b] Hence, find the equation of the tangent to the curve $y = \sin^{-1}\left(\frac{x}{2}\right)$ at the point $\left(1, \frac{\pi}{6}\right)$.

$f'(1) = \frac{1}{\sqrt{4-1}}$

$= \frac{1}{\sqrt{3}}$

$y - \frac{\pi}{6} = \frac{1}{\sqrt{3}}(x - 1)$

$\sqrt{3}y - \frac{\pi\sqrt{3}}{6} = x - 1$

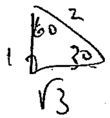
$6x - 6\sqrt{3}y + \pi\sqrt{3} - 6 = 0$

$6y - 2\sqrt{3}x + 2\sqrt{3} - \pi = 0$

[5] Find $\frac{d}{dx}\left(x\tan^{-1}x-\frac{1}{2}\ln(1+x^2)\right)$. Hence evaluate $\int_{\frac{1}{\sqrt{3}}}^{1}\tan^{-1}x\,dx$ correct to 2 decimal places.

$\frac{d}{dx} = \tan^{-1}x + \frac{x}{1+x^2} - \frac{1}{2}\times\frac{2x}{1+x^2}$

$= \tan^{-1}x$



$\int_{\frac{1}{\sqrt{3}}}^{1}\tan^{-1}x\,dx = \left[x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)\right]_{\frac{1}{\sqrt{3}}}^{1}$

$= 1\times\frac{\pi}{4} - \frac{1}{2}\ln(2) - \left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}} - \frac{1}{2}\ln\left(\frac{4}{3}\right)\right)$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2 - \frac{\pi}{\sqrt{3}\,6} + \frac{1}{2}\ln\frac{4}{3}$

$= \frac{\pi}{4} - \frac{\pi}{6\sqrt{3}} + \frac{1}{2}\ln\left(\frac{4}{3}\div 2\right)$

$= \frac{\pi}{4} - \frac{\sqrt{3}\pi}{18} + \frac{1}{2}\ln\frac{2}{3}$

$\boxed{\frac{\pi}{4}-\frac{\sqrt{3}\,\pi}{18}+\frac{1}{2}\ln\frac{2}{3}}$

[6] Find :

[a] $\int_{\frac{3}{8}}^{\frac{3}{4}}\frac{1}{\sqrt{9-16x^2}}\,dx = \int_{\frac{3}{8}}^{\frac{3}{4}}\frac{1}{\sqrt{16\left(\frac{9}{16}-x^2\right)}}\,dx$

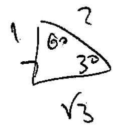
$= \frac{1}{4}\left[\sin^{-1}\frac{4}{3}x\right]_{\frac{3}{8}}^{\frac{3}{4}}$

$= \frac{1}{4}\left[\sin^{-1}1 - \sin^{-1}\frac{1}{2}\right]$

$= \frac{1}{4}\left[\frac{\pi}{2} - \frac{\pi}{6}\right]$

$= \frac{\pi}{12}.$

$\boxed{\frac{\pi}{12}}$



[b] $\int_{\frac{1}{5}}^{\frac{\sqrt{3}}{5}}\frac{1}{1+25x^2}\,dx = \int_{\frac{1}{5}}^{\frac{\sqrt{3}}{5}}\frac{1}{25\left(\frac{1}{25}+x^2\right)}\,dx$

$= \frac{1}{25}\times 5\left[\tan^{-1}5x\right]_{\frac{1}{5}}^{\frac{\sqrt{3}}{5}}$

$= \frac{1}{5}\left[\tan^{-1}\sqrt{3} - \tan^{-1}1\right]$

$= \frac{1}{5}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \frac{\pi}{60}$

$\boxed{\frac{\pi}{60}}$

[7] Differentiate with respect to x :

[a] $\ln(\sin^{-1}2x)$

$$y' = \frac{\frac{1}{\sqrt{1-4x^2}} \times 2}{\sin^{-1}2x}$$

$$= \frac{2}{\sin^{-1}2x\sqrt{1-4x^2}}$$

$$\boxed{\frac{2}{\sin^{-1}2x\sqrt{1-4x^2}}}$$

[b] $e^{\tan^{-1}\frac{x}{2}}$

$$\frac{d}{dx}\left(e^{\tan^{-1}\frac{x}{2}}\right) = \frac{1}{1+\frac{x^2}{4}} \times \frac{1}{2} \times e^{\tan^{-1}\frac{x}{2}}$$

$$= \frac{2}{4+x^2}e^{\tan^{-1}\frac{x}{2}}$$

$$\boxed{\frac{2e^{\tan^{-1}\frac{x}{2}}}{4+x^2}}$$

[8] [a] Show that if $0 < \alpha < 1$ and $0 < \beta < 1$, then

$$\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

$A = \tan^{-1}\alpha$, $B = \tan^{-1}\beta$

$\tan A = \alpha$, $\tan B = \beta$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

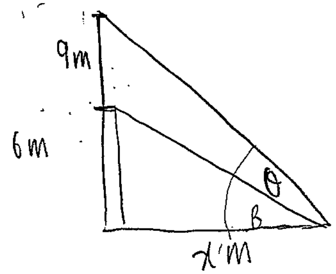
$$= \frac{\alpha+\beta}{1-\alpha\beta}$$

$$A+B = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

$$\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

[b] A flag-post 9 metres high stands on the top of a support, the top of which is 16 metres above the eye-level of an observer. If the flag-pole subtends an angle of θ at the eye of the observer who is standing x metres from the support, prove (with the aid of a diagram) that :

$$\theta = \tan^{-1}\left(\frac{9x}{x^2+400}\right).$$



$\tan(\theta+\beta) = \frac{25}{x}$ $\quad \tan\beta = \frac{16}{x}$

$$\frac{25}{x} = \frac{\tan\theta + \frac{16}{x}}{1 - \frac{16}{x}\tan\theta}$$

$$\frac{25}{x} - \frac{400}{x^2}\tan\theta = \tan\theta + \frac{16}{x}$$

$$\tan\theta + \frac{400}{x^2}\tan\theta = \frac{9}{x}$$

$$x^2\tan\theta + 400\tan\theta = 9x$$

$$\tan\theta = \frac{9x}{x^2+400}$$

$$\therefore \theta = \tan^{-1}\left(\frac{9x}{x^2+400}\right)$$

[c] Hence, or otherwise, determine the distance the observer must stand from the support to give θ a maximum value.

$\theta = \tan^{-1}\left(\frac{9x}{x^2+400}\right)$

$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{9x}{x^2+400}\right)^2} \times \frac{9(x^2+400) - 9x.2x}{(x^2+400)^2}$

$= \frac{(x^2+400)^2}{(x^2+400)^2 + 81x^2} \times \frac{9(x^2+400) - 18x^2}{(x^2+400)^2}$

$= \frac{3600 - 9x^2}{x^4 \text{ - - - -}}$

when $\frac{d\theta}{dx} = 0$, $9x^2 = 3600$

$x^2 = 400$

$x = 20$

$x > 0$

x	19	20	21
$\frac{d\theta}{dx}$	+	0	−

∴ Max $x = 20$ m

$x = 20$ m