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YEAR 12 – MATHEMATICS

REVIEW BOOKLET ON INTEGRATION, AREAS & VOLUMES

Rules for primitives: $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ or $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$

1. Find the primitive functions, $F(x)$, of the following:

(a) x^3 $F(x) =$	(b) $2x$ $F(x) =$
(c) 5	(d) $12x^2$
(e) $\frac{x^4}{2}$	(f) $x^{\frac{3}{2}}$
(g) $\sqrt{x}$	(h) x^{-3}
(i) $\frac{1}{x^2}$	(j) $\frac{6}{x^3}$

2. Find the primitive functions, $F(x)$, of the following:

(a) $x^2 + 6$ $F(x) =$	(b) $4x^2 - 2x + 5$ $F(x) =$
(c) $x^2 - 6x^3$	(d) $(x+3)^6$
*(e) $5\sqrt{x} - \frac{1}{3x^2}$	*(f) $\frac{1}{(4x-1)^3}$

Answers: (1)(a) $\frac{x^4}{4} + c$ (b) $x^2 + c$ (c) $5x + c$ (d) $4x^3 + c$ (e) $\frac{x^5}{10} + c$ (f) $\frac{2x^{\frac{5}{2}}}{5} + c$
(g) $\frac{2x^{\frac{3}{2}}}{3} + c$ (h) $-\frac{1}{2x^2} + c$ (i) $-\frac{1}{x} + c$ (j) $-\frac{3}{x^2} + c$ (2) (a) $\frac{x^3}{3} + 6x + c$
(b) $\frac{4x^3}{3} - x^2 + 5x + c$ (c) $\frac{x^3}{3} - \frac{3x^4}{2} + c$ (d) $\frac{(x+3)^7}{7} + c$ (e) $\frac{10x^{\frac{3}{2}}}{3} + \frac{1}{3x} + c$
(f) $-\frac{1}{8(4x-1)^2} + c$

Rules for definite and indefinite integrals:

The definite integral given by $\int_a^b f(x)\,dx = \left[F(x)\right]_a^b$ where a, b are the lower and upper limits of the integral and $F(x)$ is its primitive function.

(1) Find the following **"indefinite integrals"**:

(a) $\int x^2(5x-2)\,dx$

(b) $\int (5x-4)^3\,dx$

(c) $\int \sqrt{8-3x}\,dx$

(2) Evaluate the following **"definite integrals"**:

(a) $\int_4^5 (2x+3)\,dx$

(b) $\int_0^3 (4x-2)\,dx$

Answers: (1)(a) $\frac{5x^4}{4}-\frac{2x^3}{3}+c$ (b) $\frac{(5x-4)^4}{20}+c$ (c) $-\frac{2(8-3x)^{\frac{3}{2}}}{9}+c$

(2)(a) 12 (b) 12

(2) (c) $\int_0^1 \left(y^3 - y\right) dy$	(d) $\int_{-1}^3 \left(4 + 2x\right) dx$
(e) $\int_{-1}^1 \left(3t^2 + 1\right) dt$	(f) $\int_0^3 x\left(3 - x\right)\, dx$

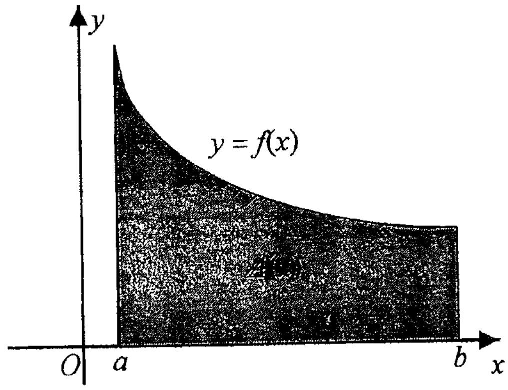
Answers: (2)(c) $-\frac{1}{4}$ (d) 24 (e) 4 (f) 4.5

Rules for finding area under the curve:

The area enclosed by the x–axis, the ordinates $x = a$, $x = b$ and the curve is given by :

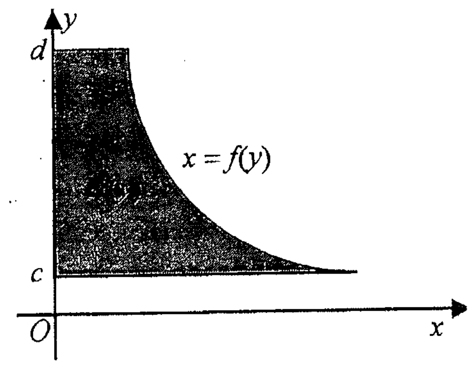
$$A(x) = \int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is a primitive function of $f(x)$.



The area enclosed by the y–axis, the lines $y = c$ and $y = d$ and the curve is given by :

$$A(y) = \int_c^d f(y)dy = F(d) - F(c)$$



Case 1 :

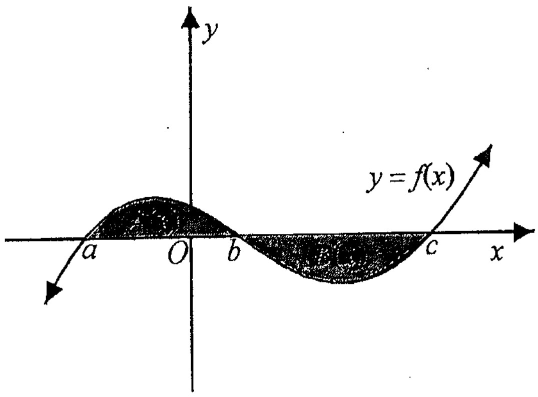
If the question clearly says to find the **area** then you must :

(a) first draw a clear diagram and

(b) then find the area enclosed as in this case to be :

$$A(x) + B(x)$$
$$= \int_a^b f(x)dx + \left| \int_b^c f(x)dx \right|$$

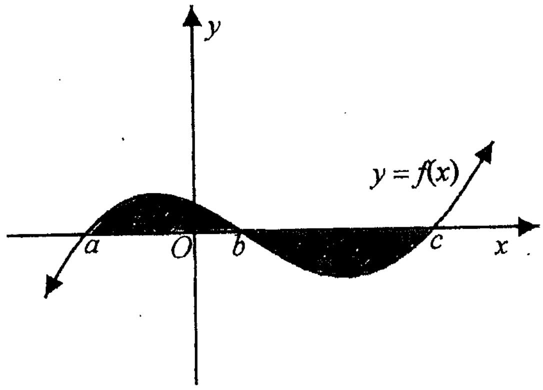
$$= \mathrm{F(b) - F(a) + \left| F(c) - F(b) \right|}$$



Case 2 :

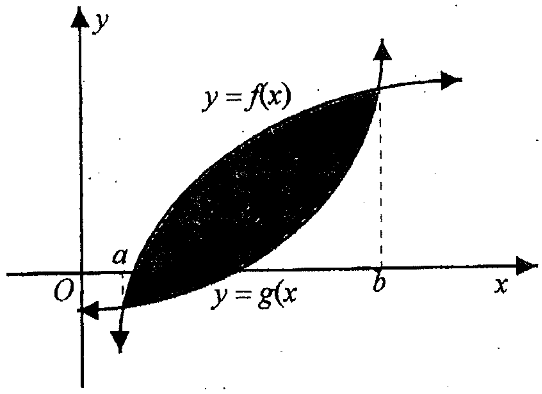
If the question only says find the integral then there is **no need** to draw a diagram, and the answer is :

$$\int_a^c f(x).dx = F(c) - F(a)$$



The area enclosed by the two curves given by $y = f(x)$ and $y = g(x)$ is

$$A(x) = \int_a^b f(x) - g(x)\, dx$$
$$= [F(b) - G(b)] - [F(a) - G(a)]$$



Exercises: For all questions concerning areas, do remember to draw a sketch.

(1) Find the **area** represented by $A = \int_1^6 \sqrt{x+3}\ dx$

Answers: (1) $12\frac{2}{3}$

(2) (a) Find the x-intercepts of $y = 4x - x^2$ and sketch the curve.

(b) Find the **area** between the curve $y = 4x - x^2$ and the x-axis between the values $x = 0$ and $x = 6$.

Answers: (2)(a) Graph (b) $21\frac{1}{3}$

(3) (a) Find the points of intersection of the curves: $y = x^2 - 2x$ and $y = 6 - x$.

(b) Use this information to find the **area** enclosed by the two curves.

Answers: (3)(a) $(-2,8),(3,3)$ Graphs (b) $20\frac{5}{6}$

Rules for finding volumes:

The volume of a solid rotated around the:

(I) x-axis is given by $\pi\int_a^b y^2 dx = \pi\int_a^b [f(x)]^2\, dx$

(II) y-axis is given by $\pi\int_c^d x^2 dy = \pi\int_c^d [f(y)]^2\, dy$

Exercises:

(1) Find the volume of revolution of the line $y = 3x$, when rotated around the x-axis between the values of $x = 0$ and $x = 5$.

(2) Find the volume generated when the curve $y = \sqrt{16 - x^2}$ is rotated around the x-axis between the values of $x = 0$ and $x = 2$.

Answers: (1) 375π units^3 (2) $\dfrac{88\pi}{3}$ units^3.

(3) Find the volume generated when the curve $y = x^2 - 1$ is rotated about the y-axis between the points $(1,0)$ and $(2,3)$.

(4) The region enclosed between the curve $y = \sqrt{16 - x^2}$ and the x axis between $x = -2$ and $x = 2$ is rotated through four right angles about the x axis. Show that the volume of the solid generated is $\frac{176\pi}{3}$.

Answers: (3) $\frac{15\pi}{2}$ units^3

Rules for approximating areas using:

(I) Trapezoidal rule : $A \approx \frac{h}{2}\left[y_0 + y_n + 2\left(y_1 + y_2 + ... + y_{n-1}\right)\right]$

(II) Simpson's rule : $A \approx \frac{h}{3}\left[y_0 + y_n + 4\left(y_1 + y_3 + ... + y_{2n-1}\right) + 2\left(y_2 + y_4 + ... + y_{2n}\right)\right]$

Exercises:

(1) (a) Use Trapezoidal rule with ***three*** function values to approximate the area under the curve $y = x^2 + x$ between the values of $x = 0$ and $x = 2$.

(b) Find the exact area given by $\int_0^2 (x^2 + x)\,dx$ and find the percentage error (to the nearest percent) in part (a) of this question.

Answers: (1)(a) 5 units^2 (b) $4\frac{2}{3}$ units^2; 7% error

(2) (a) Use Trapezoidal rule with ***four*** equal strips (subintervals) to approximate the area under the curve $y=\dfrac{1}{\sqrt{1+x^2}}$ between $x=0$ and $x=1$. (Ans to 2 d.p.)

(b) Apply Simpson's rule to part (a) and estimate the percentage error between the two methods. (i.e. Error/Trapezoidal value x 100%)

(3) Use Simpson's rule with ***seven*** function values to approximate $\int_1^4 f(x)\,dx$

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	0	0.4	0.7	0.9	1.1	1.25	1.4

<u>Answers:</u> (2)(a) 0.88 units^2 (b) 0.79 units^2;10% error (3) $2\frac{8}{15}$ units^2