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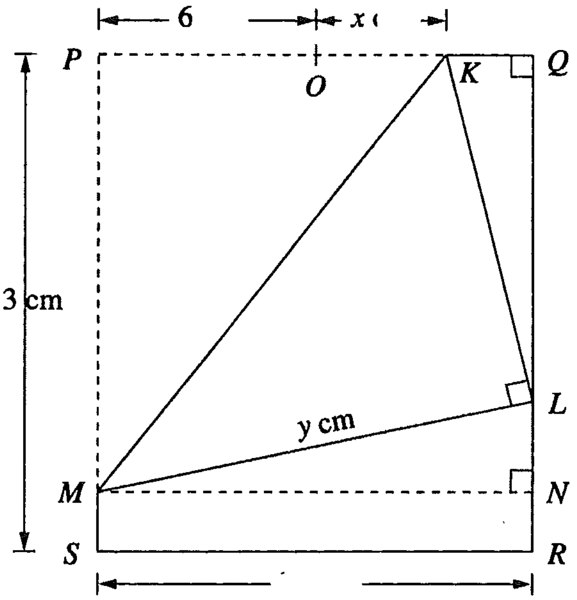
YEAR 12 – ADVANCED MATHS

REVIEW TOPIC (SP1)
PROBLEMS INVOLVING MAXIMA & MINIMA

HSC 06

*(10)

(b)



NOT TO SCALE

A rectangular piece of paper $PQRS$ has sides $PQ = 12$ cm and $PS = 13$ cm. The point O is the midpoint of PQ. The points K and M are to be chosen on OQ and PS respectively, so that when the paper is folded along KM, the corner that was at P lands on the edge QR at L. Let $OK = x$ cm and $LM = y$ cm.

Copy or trace the diagram into your writing booklet.

(i) Show that $QL^2 = 24x$. **1**

(ii) Let N be the point on QR for which MN is perpendicular to QR. **3**

By showing that $\Delta QKL \parallel\!| \Delta NLM$, deduce that $y = \dfrac{\sqrt{6}(6+x)}{\sqrt{x}}$.

(iii) Show that the area, A, of ΔKLM is given by $A = \dfrac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$. **1**

(iv) Use the fact that $12 \le y \le 13$ to find the possible values of x. **2**

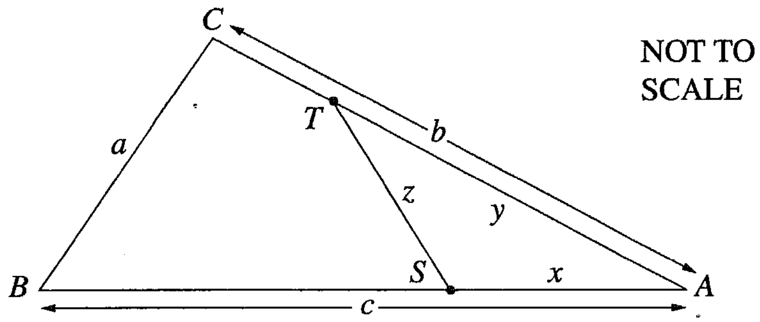
$\frac{8}{3} \le x \le 6$

(v) Find the minimum possible area of ΔKLM. **3**

$56\frac{1}{3}\text{ units}^2$

***HSC 04**
(10)

(b)



The diagram shows a triangular piece of land ABC with dimensions $AB = c$ metres, $AC = b$ metres and $BC = a$ metres, where $a \le b \le c$.

The owner of the land wants to build a straight fence to divide the land into two pieces of equal area. Let S and T be points on AB and AC respectively so that ST divides the land into two pieces of equal area.

Let $AS = x$ metres, $AT = y$ metres and $ST = z$ metres.

(i) Show that $xy = \frac{1}{2}bc$. **1**

(ii) Use the cosine rule in triangle AST to show that **2**

$$z^2 = x^2 + \frac{b^2c^2}{4x^2} - bc\cos A.$$

(iii) Show that the value of z^2 in the equation in part (ii) is a minimum when $x = \sqrt{\dfrac{bc}{2}}$. **4**

(iv) Show that the minimum length of the fence is $\sqrt{\dfrac{(P-2b)(P-2c)}{2}}$ metres, where $P=a+b+c$. **2**

(You may assume that the value of x given in part (iii) is feasible.)

HSC 03

(9)

(c) A fish is swimming at a constant speed against the current. The current is moving at a constant speed of u m s^{-1}. The speed of the fish relative to the water is v m s^{-1}, so that the actual speed of the fish is $(v-u)$ m s^{-1}.

The rate at which the fish uses energy is proportional to v^3, so the amount of energy used in t seconds is given by

$$E = av^3t,$$

where a is a constant.

(i) Show that the energy used to swim L metres is given by **1**

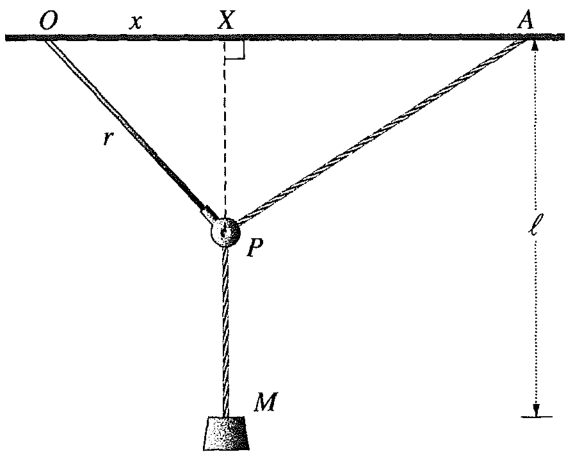
$$E = \frac{aLv^3}{v-u}.$$

(ii) Migrating fish try to minimise the total energy used to swim a fixed distance. Find the value of v that minimises E. (You may assume $v > u > 0$.) **4**

$$\boxed{v = \frac{3u}{2}}$$

*(10)

(b) A pulley P is attached to the ceiling at O by a piece of metal that can swing freely. One end of a rope is attached to the ceiling at A. The rope is passed through the pulley P and a weight is attached to the other end of the rope at M, as shown in the diagram.



The distance OA is 1 m, the length of the rope is 2 m, and the length of the piece of metal $OP = r$ metres, where $0 < r < 1$. Let X be the point where the line MP produced meets OA. Let $OX = x$ metres and $XM = \ell$ metres.

(i) By considering triangles OXP and AXP, show that **1**

$$\ell = 2 + \sqrt{r^2 - x^2} - \sqrt{1 - 2x + r^2} .$$

(ii) Show that $\dfrac{d\ell}{dx} = \dfrac{\left(r^2 - x^2\right) - x^2\left(1 - 2x + r^2\right)}{\sqrt{r^2 - x^2}\sqrt{1 - 2x + r^2}\left(\sqrt{r^2 - x^2} + x\sqrt{1 - 2x + r^2}\right)}$. **2**

(iii) You are given the factorisation **2**

$$\left(r^2 - x^2\right) - x^2\left(1 - 2x + r^2\right) = (x-1)\left(2x^2 - r^2x - r^2\right).$$

(Do NOT prove this.)

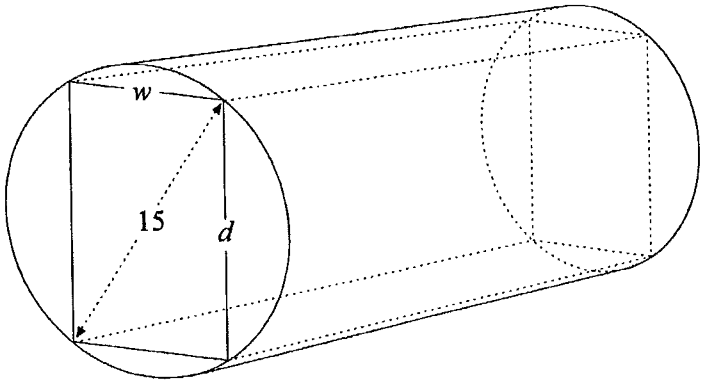
Find the value of x for which M is closest to the floor. Justify your answer.

$$x = \frac{r\left(r + \sqrt{r^2 + 8}\right)}{4}$$

HSC 95

(9)

(a) **7**



A rectangular beam of width w cm and depth d cm is cut from a cylindrical pine log as shown.

The diameter of the cross-section of the log (and hence the diagonal of the cross-section of the beam) is 15 cm.

The strength S of the beam is proportional to the product of its width and the square of its depth, so that

$$S = kd^2w.$$

(i) Show that $S = k\left(225w - w^3\right)$.

(ii) What numerical dimensions will give a beam of maximum strength? Justify your answer.

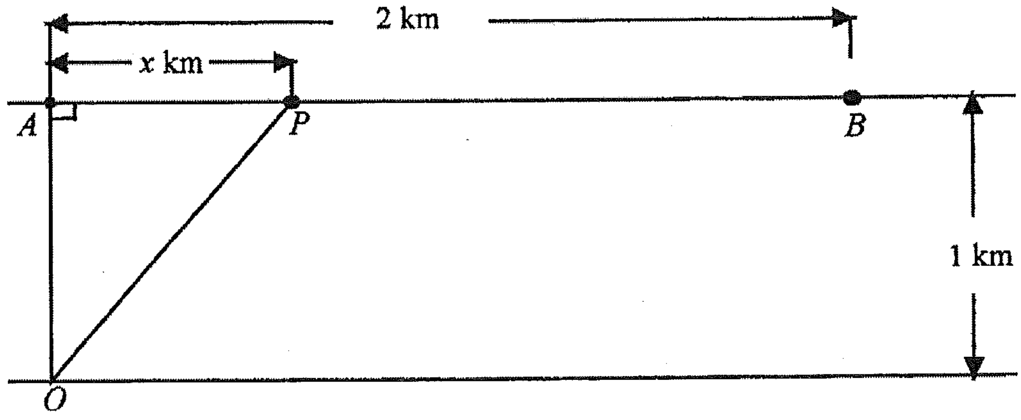
$w = 5\sqrt{3}\,; d = 5\sqrt{6}$

(iii) A square beam with diagonal 15 cm could have been cut from the log. Show that the rectangular beam of maximum strength is more than 8% stronger than this square beam.

Show that $S_{\text{rect}} = 1299k$; $S_{\text{square}} = 1193k$; % difference $= 8.9\%$

HSC '91

(10)(b)



The diagram shows a straight section of a river, one kilometre wide. Adrienna is at a point O on one bank and she wishes to reach a point B on the opposite bank. The point A is directly opposite O and the distance from A to B is two kilometres.

Adrienna can row at 6 km/h and jog at 10 km/h. She intends to row in a straight line to point P on the opposite bank and then jog directly from P to B.

Let the distance AP be x kilometres.

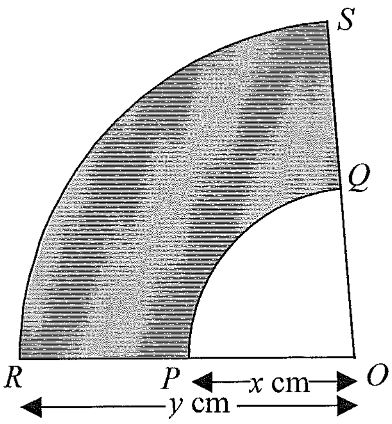
(i) Show that the time T, in hours, that Adrienna takes to reach B is given by

$$T = \frac{\sqrt{x^2+1}}{6} + \frac{2-x}{10}$$

(ii) Show that if Adrienna wishes to minimise the time taken to complete the journey then she should row to a point P, $\frac{3}{4}$ kilometre from A.

Further Exercises in Max / Min

(1)



In the diagram where PQ and RS are arcs of concentric circles centre O, $OP = x$ cm and $OR = y$ cm. The angle $POQ = 1\frac{1}{2}$ radians.

The area of the shaded region is A cm^2.

(a) Show that the expression for A in terms of x and y is $A = \frac{3}{4}\left(y^2 - x^2\right)$.

(b) Given that the perimeter of the shaded region $PQSR$ is 80 cm, find the value of x for which A is a maximum.

$x = 3\frac{1}{3}$ cm

*(2) Two men John and Bob start walking along two roads towards the intersection of the roads which meet at right angles.
John start at 9 km from the intersection and walks at 4 km/hr and Bob starts at 13 km from the intersection and walks at 3 km/hr. With the aid of a diagram,

(a) Show that their distance apart x is $\sqrt{25t^2 - 150t + 250}$.

(b) Find the time taken for them to be the shortest distance apart.

$\boxed{t = 3 \text{ hr}}$