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# YEAR 12 – ADVANCED MATHS

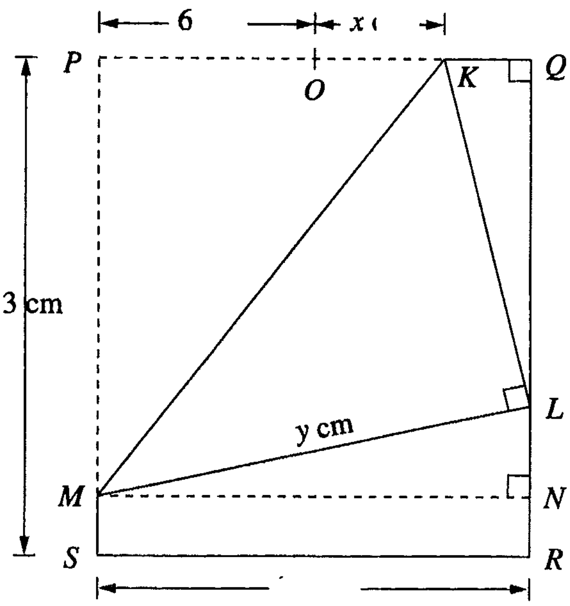
# REVIEW TOPIC (SP1)

# PROBLEMS INVOLVING MAXIMA & MINIMA

**PAST HSC EXAMINATION QUESTIONS: (* indicates difficult questions and to be attempted last)**
**HSC 06**

*(10)

(b)




NOT TO SCALE

A rectangular piece of paper $PQRS$ has sides $PQ = 12$ cm and $PS = 13$ cm. The point $O$ is the midpoint of $PQ$. The points $K$ and $M$ are to be chosen on $OQ$ and $PS$ respectively, so that when the paper is folded along $KM$, the corner that was at $P$ lands on the edge $QR$ at $L$. Let $OK = x$ cm and $LM = y$ cm.

Copy or trace the diagram into your writing booklet.

(i) Show that $QL^2 = 24x$. 1

(ii) Let $N$ be the point on $QR$ for which $MN$ is perpendicular to $QR$. **3**

By showing that $\Delta QKL \,|||\, \Delta NLM$, deduce that $y = \dfrac{\sqrt{6}(6+x)}{\sqrt{x}}$.

(iii) Show that the area, $A$, of $\Delta KLM$ is given by $A = \dfrac{\sqrt{6}(6+x)^2}{2\sqrt{x}}$. **1**

**(iv) Use the fact that $12 \le y \le 13$ to find the possible values of $x$.** **2**

$$\boxed{\frac{8}{3} \le x \le 6}$$

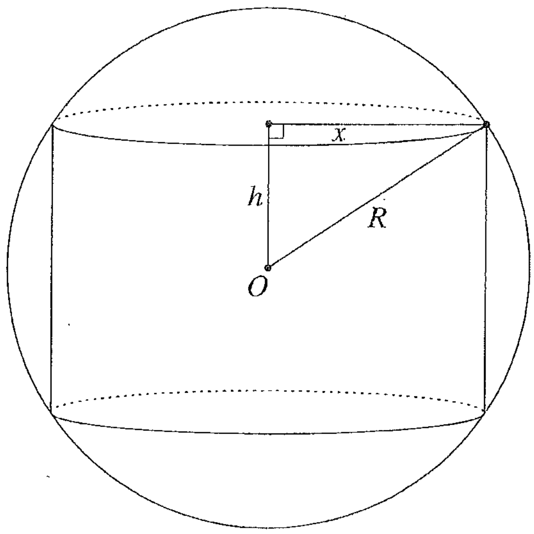
**(v) Find the minimum possible area of $\Delta KLM$.** **3**

$56\frac{1}{3}$ units$^2$

**HSC 05**

(8)

(a)




A cylinder of radius $x$ and height $2h$ is to be inscribed in a sphere of radius $R$ centred at $O$ as shown.

(i) Show that the volume of the cylinder is given by **1**

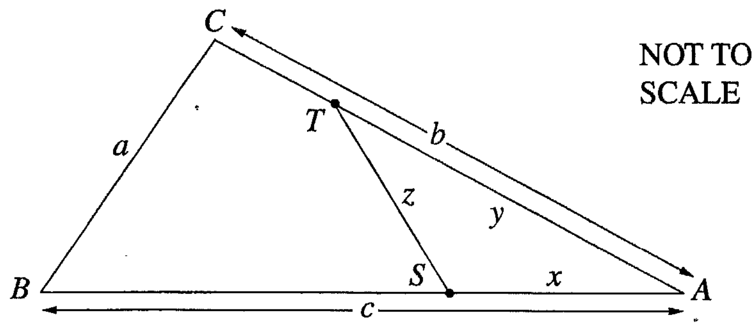
$$V = 2\pi h(R^2 - h^2).$$

(ii) Hence, or otherwise, show that the cylinder has a maximum volume when $h = \dfrac{R}{\sqrt{3}}$. **3**

***HSC 04**

(10)

(b)




The diagram shows a triangular piece of land $ABC$ with dimensions $AB = c$ metres, $AC = b$ metres and $BC = a$ metres, where $a \le b \le c$.

The owner of the land wants to build a straight fence to divide the land into two pieces of equal area. Let $S$ and $T$ be points on $AB$ and $AC$ respectively so that $ST$ divides the land into two pieces of equal area.

Let $AS = x$ metres, $AT = y$ metres and $ST = z$ metres.

(i) Show that $xy = \frac{1}{2}bc$. **1**

(ii) Use the cosine rule in triangle $AST$ to show that **2**

$$z^2 = x^2 + \frac{b^2c^2}{4x^2} - bc\cos A.$$

(iii) Show that the value of $z^2$ in the equation in part (ii) is a minimum when $x = \sqrt{\dfrac{bc}{2}}$. **4**

(iv) Show that the minimum length of the fence is $\sqrt{\dfrac{(P-2b)(P-2c)}{2}}$ metres, **2**

where $P=a+b+c$.

(You may assume that the value of $x$ given in part (iii) is feasible.)

**HSC 03**

(9)

(c) A fish is swimming at a constant speed against the current. The current is moving at a constant speed of $u\ \text{m s}^{-1}$. The speed of the fish relative to the water is $v\ \text{m s}^{-1}$, so that the actual speed of the fish is $(v-u)\ \text{m s}^{-1}$.

The rate at which the fish uses energy is proportional to $v^3$, so the amount of energy used in $t$ seconds is given by

$$E = av^3t,$$

where $a$ is a constant.

(i) Show that the energy used to swim $L$ metres is given by **1**

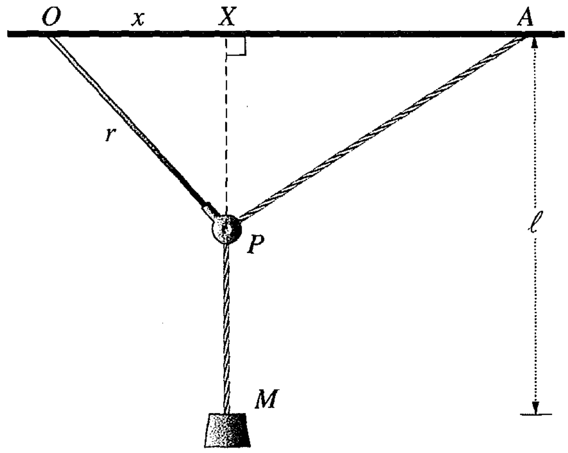
$$E = \frac{aLv^3}{v-u}.$$

(ii) Migrating fish try to minimise the total energy used to swim a fixed distance. Find the value of $v$ that minimises $E$. (You may assume $v > u > 0$.) **4**

$$\boxed{v = \frac{3u}{2}}$$

*(10)

(b) A pulley $P$ is attached to the ceiling at $O$ by a piece of metal that can swing freely. One end of a rope is attached to the ceiling at $A$. The rope is passed through the pulley $P$ and a weight is attached to the other end of the rope at $M$, as shown in the diagram.




The distance $OA$ is 1 m, the length of the rope is 2 m, and the length of the piece of metal $OP = r$ metres, where $0 < r < 1$. Let $X$ be the point where the line $MP$ produced meets $OA$. Let $OX = x$ metres and $XM = \ell$ metres.

(i) By considering triangles $OXP$ and $AXP$, show that **1**

$$\ell = 2 + \sqrt{r^2 - x^2} - \sqrt{1 - 2x + r^2}\,.$$

(ii) Show that $\dfrac{d\ell}{dx} = \dfrac{\left(r^2 - x^2\right) - x^2\left(1 - 2x + r^2\right)}{\sqrt{r^2 - x^2}\sqrt{1 - 2x + r^2}\left(\sqrt{r^2 - x^2} + x\sqrt{1 - 2x + r^2}\right)}$. **2**

(iii) You are given the factorisation **2**

$$\left(r^2 - x^2\right) - x^2\left(1 - 2x + r^2\right) = (x-1)\left(2x^2 - r^2x - r^2\right).$$

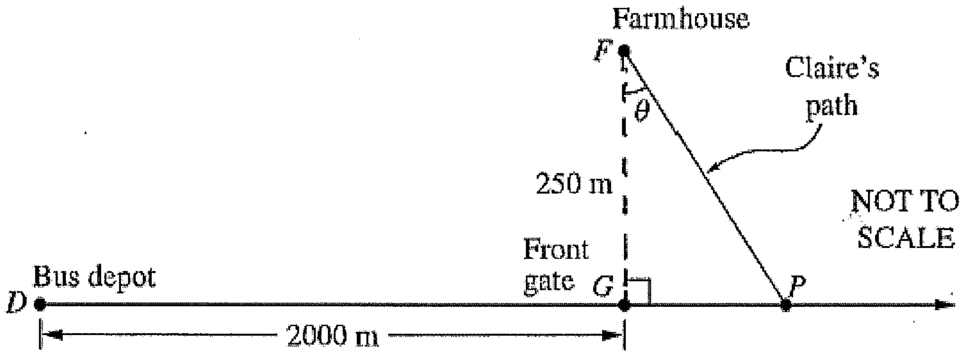
(Do NOT prove this.)

Find the value of $x$ for which $M$ is closest to the floor. Justify your answer.

$$x = \frac{r\left(r + \sqrt{r^2 + 8}\right)}{4}$$

**HSC 01**
*(10)

(b)




The diagram shows a farmhouse $F$ that is located 250 m from a straight section of road. The road begins at the bus depot $D$, which is situated 2000 m from the front gate $G$ of the farmhouse. The school bus leaves the depot at 8 am and travels along the road at a speed of 15 m s$^{-1}$. Claire lives in the farmhouse, and she can run across the open paddock between the house and the road at a speed of 4 m s$^{-1}$. The bus will stop for Claire anywhere on the road, but will not wait for her.

Assume that Claire catches the bus at the point $P$ on the road where $\angle GFP = \theta$.

(i) Find two expressions in terms of $\theta$, one expression for the time taken for the bus to travel from $D$ to $P$ and the other expression for the time taken by Claire to run from $F$ to $P$. **2**

$$T_{DP} = \frac{5(8+\tan\theta)}{18}\text{mins; } T_{FP} = \frac{25\sec\theta}{24}\text{mins}$$

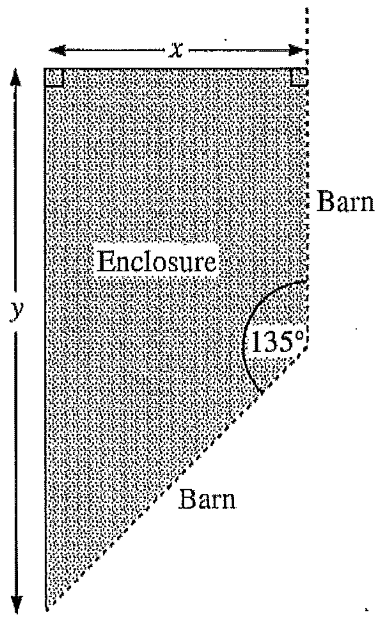
(ii) What is the latest time that Claire can leave home in order to catch the bus? **4**

1min 13sec past 8am

**HSC 2000**

(8)

(b) An enclosure is to be built adjoining a barn, as in the diagram. The walls of the barn meet at 135°, and 117 metres of fencing is available for the enclosure, so that $x + y = 117$ where $x$ and $y$ are as shown in the diagram. **5**



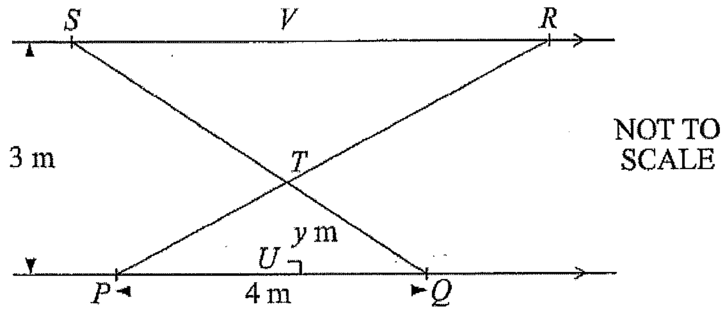

(i) Show that the shaded area of the enclosure in square metres is given by

$$A = 117x - \frac{3}{2}x^2 .$$

(ii) Show that the largest area of the enclosure occurs when $y = 2x$.

**HSC '99**

(9)(b) 9




In the diagram, $PQ$ and $SR$ are parallel railings which are 3 metres apart. The points $P$ and $Q$ are fixed 4 metres apart on the lower railing. Two crossbars $PR$ and $QS$ intersect at $T$ as shown in the diagram. The line through $T$ perpendicular to $PQ$ intersects $PQ$ at $U$ and $SR$ at $V$. The length of $UT$ is $y$ metres.

(i) By using similar triangles, or otherwise, show that $\dfrac{SR}{PQ} = \dfrac{VT}{UT}$.

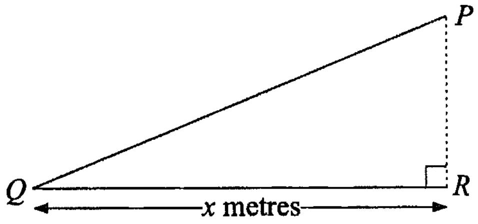
(ii) Show that $SR = \dfrac{12}{y} - 4$.

(iii) Hence show that the total area $A$ of $\Delta PTQ$ and $\Delta RTS$ is $A = 4y - 12 + \frac{18}{y}$.

(iv) Find the value of $y$ that minimises $A$. Justify your answer.

**HSC 97**
(6)

(a) **6**




A wire of length 5 metres is to be bent to form the hypotenuse and base of a right-angled triangle $PQR$, as shown in the diagram. Let the length of the base $QR$ be $x$ metres.

(i) What is the length of the hypotenuse $PQ$ in terms of $x$?

$\boxed{5-x}$

(ii) Show that the area of the triangle $PQR$ is $\frac{1}{2}x\sqrt{25-10x}$ square metres.

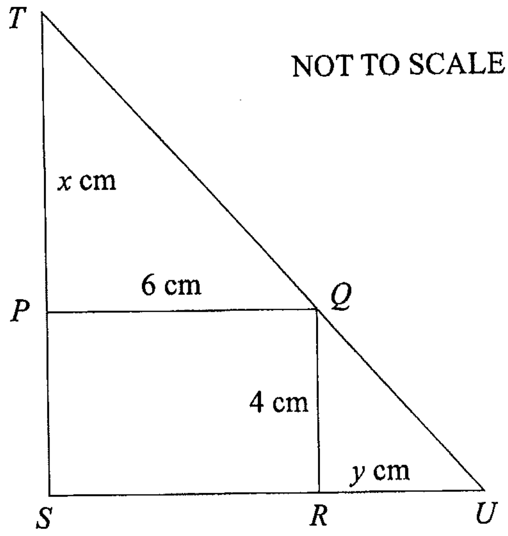
**(iii) What is the maximum possible area of the triangle?**

$$\boxed{\frac{25\sqrt{3}}{18} \approx 2.41 \text{ m}^2 \text{ (to 2 d.p.)}}$$

**HSC 96**

(8)

(b) **8**




*PQRS* is a rectangle with $PQ = 6$ cm and $QR = 4$ cm. *T* and *U* lie on the lines *SP* and *SR* respectively, so that *T*, *Q*, and *U* are collinear, as shown in the diagram. Let $PT = x$ cm and $RU = y$ cm.

(i) Show that triangles *TPQ* and *QRU* are similar.

(ii) Show that $xy = 24$.

(iii) Show that the area, $A$, of triangle $TSU$ is given by

$$A = 24 + 3x + \frac{48}{x}.$$

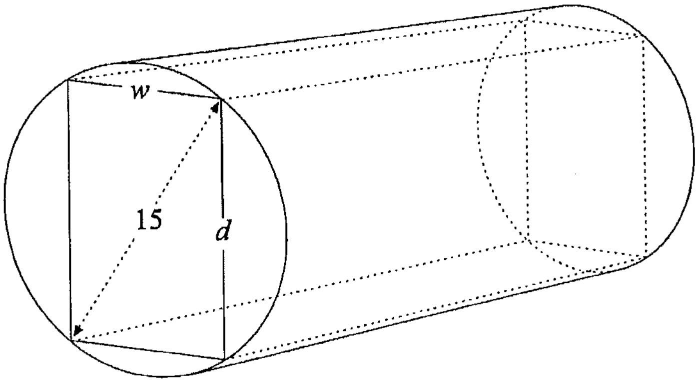
(iv) Find the height and base of the triangle *TSU* with minimum area. Justify your answer.

$TS = 8$ cm; $SU = 12$ cm

**HSC 95**

(9)

(a) **7**




A rectangular beam of width $w$ cm and depth $d$ cm is cut from a cylindrical pine log as shown.

The diameter of the cross-section of the log (and hence the diagonal of the cross-section of the beam) is 15 cm.

The strength $S$ of the beam is proportional to the product of its width and the square of its depth, so that

$$S = kd^2w.$$

(i) Show that $S = k(225w - w^3)$.

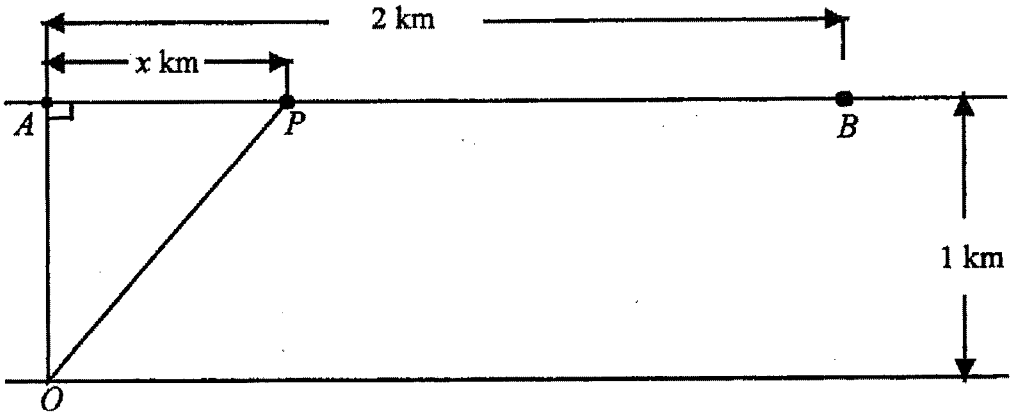
(ii) What numerical dimensions will give a beam of maximum strength? Justify your answer.

$w = 5\sqrt{3}\,; d = 5\sqrt{6}$

(iii) A square beam with diagonal 15 cm could have been cut from the log. Show that the rectangular beam of maximum strength is more than 8% stronger than this square beam.

Show that $S_{\text{rect}} = 1299k$; $S_{\text{square}} = 1193k$; % difference $= 8.9\%$

**HSC '91**
(10)(b)




The diagram shows a straight section of a river, one kilometre wide. Adrienna is at a point $O$ on one bank and she wishes to reach a point $B$ on the opposite bank. The point $A$ is directly opposite $O$ and the distance from $A$ to $B$ is two kilometres.

Adrienna can row at 6 km/h and jog at 10 km/h. She intends to row in a straight line to point $P$ on the opposite bank and then jog directly from $P$ to $B$.

Let the distance $AP$ be $x$ kilometres.

(i) Show that the time $T$, in hours, that Adrienna takes to reach $B$ is given by

$$T = \frac{\sqrt{x^2+1}}{6} + \frac{2-x}{10}$$

(ii) Show that if Adrienna wishes to minimise the time taken to complete the journey then she should row to a point $P$, $\frac{3}{4}$ kilometre from $A$.