C.E.M.TUITION

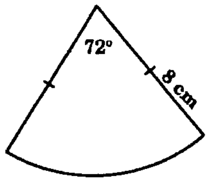
Name : ______________________________

Review Topic : Trigonometric Functions

(HSC - PAPER 1)

Year 12 - 2 Unit

1. (a)



The figure shows a sector of a circle with radius 8 cm. Find the area of this sector correct to 2 decimal places.

(b) Differentiate: **(i)** $\log_e(\sin x)$ **(ii)** $\cos\left(x+\frac{\pi}{2}\right)$

(c) Evaluate: **(i)** $\int_0^{\frac{\pi}{4}} \cos 2x\, dx$ **(ii)** $\int_0^1 \sin\frac{\pi}{3}x\, dx$

(d) Find the equation of the tangent to $y = 2\sin 2x$ at the point $x = \frac{\pi}{12}$.

2. **(a)** Solve, where $0 \le x \le 2\pi$, $2\sin x = \sqrt{3}$.

(b) Evaluate $\lim_{\theta \to 0} \frac{\sin 3\theta}{\theta}$.

(c) **(i)** Sketch the graph of $y = \cos x$, for $0 \le x \le \pi$.

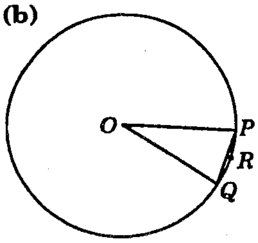
(ii) Where does the curve cross the x axis?

(iii) Show that the area between the curve and the x axis between $x = 0$ and $x = \pi$ is 2 units^2.

(d) Find $\frac{dy}{dx}$ if: **(i)** $y = \tan\frac{x}{3}$ **(ii)** $y = \sin^2 x$

3. **(a)** Find the gradient of the normal to $y = \cos 3x$ at the point $x = \frac{\pi}{6}$.

(b)



The figure shows a circle centre O with triangle OPQ and R is a point on arc PQ. $OP = 10$ cm and $\angle POQ = 30°$.

(i) Find the area of ΔOPQ.

(ii) Express 30° in radians and show that the area of sector OPQ is $\frac{25\pi}{3}$ cm^2.

(iii) Show that the area of the segment PRQ is $25\left(\frac{\pi - 3}{3}\right)$ cm^2.

(iv) Use the Cosine Rule to find the length of PQ.

(v) Show that the length PQ is shorter than the arc PRQ by 0·06 cm, correct to 2 dec. places.

(c) Find the nature of the stationary points on $y = \sin 2x$ if $0 < x < \pi$ (using calculus).

(d) Find c, if $\int_0^c \cos x \, dx = \frac{\sqrt{3}}{2}$, where $0 < c < \frac{\pi}{2}$.

4. (a) Differentiate: **(i)** $\dfrac{\sin x}{x}$ **(ii)** $e^{2\cos x}$

(b) Solve, for $0° \le x \le 360°$, $2\sin x - 1 = 0$

(c) Evaluate: **(i)** $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$ **(ii)** $\displaystyle\int_0^{\frac{\pi}{2}} \sin x + \cos x \, dx$

(d) Prove that $\dfrac{d}{dx}\left[\dfrac{1+\sin x}{\cos x}\right] = \dfrac{1}{1-\sin x}$.

5. **(a)** Evaluate $\lim_{x \to 0} \frac{\sin 2x}{4x}$.

(b) **(i)** Sketch $y = \sin \pi x$, on a number plane, where $-2 \le x \le 2$.

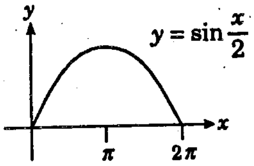
(ii) State the period and amplitude of $y = \sin \pi x$.

(iii) Use the sketch to solve the equation $\sin \pi x = 0$.

(c) If $y = 4\sin 3x$, prove that $y'' + 9y = 0$.

(d) **(i)** Show that $\int_0^{\pi} \sin \frac{x}{2}\, dx = 2$.

(ii)



Find the area between the curve $y = \sin \frac{x}{2}$ and the x axis when $x = 0$ and $x = 2\pi$.

1. (a) $A = \frac{1}{2}r^2\theta$

Note θ must be in radians.

72 degrees $= 72 \times \frac{\pi}{180}$

$= 1{\cdot}256\ 6371$ (from calc.)

$\therefore A = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 8^2 \times 1{\cdot}256\ 637\ 1$

$= 40{\cdot}212\ 386$

$= 40{\cdot}21$ (2 dec. places)

$\therefore$ area is $40{\cdot}21$ cm^2.

(b) (i) $\frac{d}{dx}[\log_e(\sin x)] = \frac{\cos x}{\sin x}$

$\boxed{\frac{d}{dx}[\log f(x)] = \frac{f'(x)}{f(x)}}$

$= \cot x$.

Note $\frac{\cos x}{\sin x} = \cot x$

(ii) $\frac{d}{dx}\left[\cos\left(x + \frac{\pi}{2}\right)\right]$

$= -\sin\left(x + \frac{\pi}{2}\right)$

(c) (i) $\int_0^{\frac{\pi}{4}} \cos 2x\, dx$

$= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{4}\right) - \sin 2(0)\right]$

$= \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin 0\right]$

$= \frac{1}{2}[1 - 0]$

$= \frac{1}{2}$.

(ii) $\int_0^1 \sin\frac{\pi}{3}x\, dx = \left[-\frac{3}{\pi}\cos\frac{\pi}{3}x\right]_0^1$

$= -\frac{3}{\pi}\left[\cos\frac{\pi}{3}(1) - \cos\frac{\pi}{3}(0)\right]$

$= -\frac{3}{\pi}\left[\cos\frac{\pi}{3} - \cos 0\right]$

$= -\frac{3}{\pi}\left[\frac{1}{2} - 1\right]$

$= -\frac{3}{\pi}\left[-\frac{1}{2}\right]$

$= \frac{3}{2\pi}$.

(d) $y = 2\sin 2x$

Subs. $x = \frac{\pi}{12}$ in $y = 2\sin 2x$

$= 2\sin 2\left(\frac{\pi}{12}\right)$

$= 2\sin\frac{\pi}{6}$

$= 2 \times \frac{1}{2}$

$= 1$

$\therefore$ the point $\left(\frac{\pi}{12}, 1\right)$.

For gradient, $\frac{dy}{dx} = 4\cos 2x$

Subs. in $x = \frac{\pi}{12}$

$\therefore \frac{dy}{dx} = 4\cos 2\left(\frac{\pi}{12}\right)$

$= 4\cos\frac{\pi}{6}$

$= 4\left(\frac{\sqrt{3}}{2}\right)$

$= 2\sqrt{3}$.

$\therefore$ Equation of tangent:

$y - y_1 = m(x - x_1)$

$y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{12}\right)$

$y - 1 = 2\sqrt{3}x - \frac{\sqrt{3}\pi}{6}$

2. (a) $2\sin x = \sqrt{3}$

$\therefore \sin x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{3}, \frac{2\pi}{3}$

Pos. in 1st and in 2nd quad.

(b) $\lim_{\theta \to 0} \frac{\sin 3\theta}{\theta}$

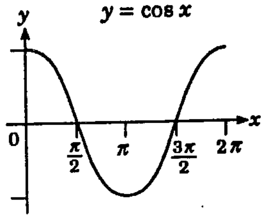
$= \lim_{\theta \to 0} \frac{\sin 3\theta}{3\theta}.3$

$= 3\lim_{\theta \to 0} \frac{\sin 3\theta}{3\theta}$

$= 3.1$ $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$

$= 3$

(c) (i)



(ii) Cut x axis when

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

(as seen from the diagram).

(iii) Area $= \int_0^{\frac{\pi}{2}} \cos x\, dx + \left|\int_{\frac{\pi}{2}}^{\pi} \cos x\, dx\right|$

$= [\sin x]_0^{\frac{\pi}{2}} + \left|[\sin x]_{\frac{\pi}{2}}^{\pi}\right|$

$= \left(\sin\frac{\pi}{2} - \sin 0\right) + \left|\sin\pi - \sin\frac{\pi}{2}\right|$

$= (1 - 0) + |-1 - 0|$

$= 1 + |-1|$

$= 1 + 1$

$= 2$.

$\therefore$ area is 2 units^2.

(d) (i) $y = \tan\frac{x}{3}$ $\boxed{\frac{x}{3} = \frac{1}{3}x}$

$\therefore y = \tan\frac{1}{3}x$

$\therefore \frac{dy}{dx} = \frac{1}{3}\sec^2\frac{1}{3}x$

$= \frac{1}{3}\sec^2\frac{x}{3}$.

(ii) $y = \sin^2 x$

$\therefore y = (\sin x)^2$

$\therefore \frac{dy}{dx} = 2(\sin x)'.\cos x$

$= 2\sin x\cos x$.

3. (a) $y = \cos 3x$

$\frac{dy}{dx} = -3\sin 3x$

Subs. $x = \frac{\pi}{6}$ in $\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = -3\sin 3\left(\frac{\pi}{6}\right)$

$= -3\sin\frac{\pi}{2}$

$= -3.1$

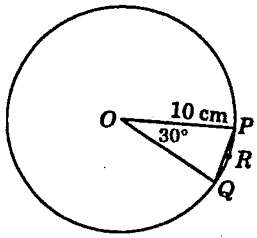
$= -3$

$\therefore$ grad. of tangent $= -3$,

$\therefore$ grad. of normal $= \frac{1}{3}$

$\therefore$ gradient of normal to $y = \cos 3x$ at $x = \frac{\pi}{6}$ is $\frac{1}{3}$.

(b)



(i) $A = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 10 \times 10 \times \sin 30$

$= \frac{1}{2} \times 10 \times 10 \times \frac{1}{2}$

$= 25$

$\therefore$ area of $\Delta\, OPQ$ is $25\ \text{cm}^2$.

(ii) $30° = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$

Deg. $\rightarrow$ Rad: $\times \frac{\pi}{180}$

$A = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} . 10^2 . \frac{\pi}{6}$

$= \frac{25\pi}{3}$

$\therefore$ area of sector OPQ is $\frac{25\pi}{3}\ \text{cm}^2$.

(iii) Area of segment = area of sector − area of triangle

$= \frac{25\pi}{3} - 25$

$= 25\left(\frac{\pi}{3} - 1\right)$

$= 25\left(\frac{\pi - 3}{3}\right)$

$\therefore$ area of segment is $25\left(\frac{\pi - 3}{3}\right)\ \text{cm}^2$.

(iv)

$PQ^2 = 10^2 + 10^2 - 2(10)(10)\cos 30°$

$a^2 = b^2 + c^2 - 2bc\cos A$

$= 100 + 100 - 200 . \frac{\sqrt{3}}{2}$

$= 200 - 100\sqrt{3}$

$= 26{\cdot}794\,919$ (from calc.)

$\therefore PQ = \sqrt{26{\cdot}794\,919}$

$= 5{\cdot}176\,380\,9$ (from calc.)

(v)

Now arc length $PRQ = r\theta$

$\ell = r\theta$

$= 10 . \frac{\pi}{6}$

$= \frac{5\pi}{3}$

$= 5{\cdot}235\,987\,8$ (from calc.)

$\therefore$ Difference

$= 5{\cdot}235\,987\,8 - 5{\cdot}176\,380\,9$

$= 0{\cdot}059\,606\,8$ (from calc.)

$= 0{\cdot}06$ (to 2 dec. pl.)

$\therefore PQ$ is shorter by 0·06 cm.

(c) $y = \sin 2x$

$\frac{dy}{dx} = 2\cos 2x = 0$

Stat. pts. when $\frac{dy}{dx} = 0$

$\therefore \cos 2x = 0$

$\therefore \quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore \quad x = \frac{\pi}{4}, \frac{3\pi}{4}$.

Subs. in $y = \sin 2x$

$\therefore x = \frac{\pi}{4} \quad \therefore y = \sin 2\left(\frac{\pi}{4}\right)$

$= \sin \frac{\pi}{2}$

$= 1$

$\therefore x = \frac{3\pi}{4} \quad \therefore y = \sin 2\left(\frac{3\pi}{4}\right)$

$= \sin \frac{3\pi}{2}$

$= -1$

$\therefore$ stat. points at

$\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right)$.

$\frac{d^2y}{dx^2} = -4\sin 2x$

Now subs. $x = \frac{\pi}{4}$ in $\frac{d^2y}{dx^2}$

$\therefore \frac{d^2y}{dx^2} = -4\sin 2\left(\frac{\pi}{4}\right)$

$= -4\sin\frac{\pi}{2}$

$= -4.1$

$= -4 < 0$

$\therefore$ max.

Also, subs. $x = \frac{3\pi}{4}$ in $\frac{d^2y}{dx^2}$

$\therefore \frac{d^2y}{dx^2} = -4\sin 2\left(\frac{3\pi}{4}\right)$

$= -4\sin\frac{3\pi}{2}$

$= -4(-1)$

$= 4 > 0$

$\therefore$ min.

$\therefore$ maximum at $\left(\frac{\pi}{4}, 1\right)$ and minimum at $\left(\frac{3\pi}{4}, -1\right)$.

(d) $\int_0^c \cos x\, dx = \frac{\sqrt{3}}{2}$

$\therefore \int_0^c \cos x\, dx = \left[\sin x\right]_0^c$

$= \sin c - \sin 0$

$= \sin c$

$\therefore \quad \sin c = \frac{\sqrt{3}}{2}$

$\therefore \quad c = \frac{\pi}{6}$.

4. (a) (i) $\frac{d}{dx}\left[\frac{\sin x}{x}\right]$

$u = \sin x$	$v = x$
$\frac{du}{dx} = \cos x$	$\frac{dv}{dx} = 1$

$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{x.\cos x - \sin x.1}{x^2}$

$= \frac{x\cos x - \sin x}{x^2}$.

(ii) $\frac{d}{dx}\left[e^{2\cos x}\right]$

$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$

$= -2\sin x e^{2\cos x}$

(b) $2\sin x - 1 = 0$

$2\sin x = 1$

$\sin x = \frac{1}{2}$

$\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$.

(c) (i) $\int_0^{\frac{\pi}{8}} \sec^2 2x\, dx$

$= \left[\frac{1}{2}\tan 2x\right]_0^{\frac{\pi}{8}}$

$= \frac{1}{2}\left[\tan 2\left(\frac{\pi}{8}\right) - \tan 2(0)\right]$

$= \frac{1}{2}\left[\tan\frac{\pi}{4} - \tan 0\right]$

$= \frac{1}{2}[1-0]$

$= \frac{1}{2}$

(ii) $\int_0^{\frac{\pi}{2}} \sin x + \cos x\, dx$

$= \left[-\cos x + \sin x\right]_0^{\frac{\pi}{2}}$

$= \left(-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - (-\cos 0 + \sin 0)$

$= (0+1) - (-1+0)$

$= 1+1$

$= 2$

(d) $\frac{d}{dx}\left[\frac{1+\sin x}{\cos x}\right]$

$u = 1+\sin x$	$v = \cos x$
$\frac{du}{dx} = \cos x$	$\frac{dv}{dx} = -\sin x$

$= \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \dfrac{\cos x.\cos x - (1+\sin x).-\sin x}{(\cos x)^2}$

$= \dfrac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$

$(\cos x)^2 = \cos^2 x$

$= \dfrac{1+\sin x}{1-\sin^2 x}$ $\quad$ $\cos^2 x = 1 - \sin^2 x$

$= \dfrac{\cancel{1+\sin x}}{(1-\sin x)(\cancel{1+\sin x})}$

$a^2 - b^2 = (a-b)(a+b)$

$= \dfrac{1}{1-\sin x}$

$\therefore \frac{d}{dx}\left[\frac{1+\sin x}{\cos x}\right] = \frac{1}{1-\sin x}.$

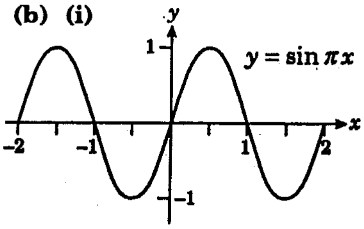
5. (a) $\lim_{x\to 0} \frac{\sin 2x}{4x}$

$= \frac{1}{2}\lim_{x\to 0}\frac{\sin 2x}{2x}$

$= \frac{1}{2}.1$ $\quad$ $\lim_{X\to 0}\frac{\sin X}{X}$

$= \frac{1}{2}$

(b) (i)



(ii) $y = \sin \pi x$

$\therefore$ period: 2

Note

Period $= \frac{2\pi}{n} = \frac{2\pi}{\pi} = 2$

amplitude: 1

(iii) For $\sin \pi x = 0$,

$\therefore$ intersection of $y = \sin \pi x$ and $y = 0$ [i.e. x axis],

$\therefore x = -2, -1, 0, 1, 2.$

(c) $y = 4\sin 3x$

$y' = 12\cos 3x$

$y'' = -36\sin 3x$

For $y'' + 9y = 0$

$\therefore$ LHS $= y'' + 9y$

$= -36\sin 3x + 9(4\sin 3x)$

$= -36\sin 3x + 36\sin 3x$

$= 0 \qquad =$ RHS

$\therefore y'' + 9y = 0$

(d) (i) $\int_0^{\pi} \sin\frac{x}{2}\, dx$

$= \int_0^{\pi} \sin\frac{1}{2}x\, dx$

$= \left[-2\cos\frac{x}{2}\right]_0^{\pi}$

$= -2\left[\cos\frac{\pi}{2} - \cos 0\right]$

$= -2(0-1)$

$= 2$

$\therefore \int_0^{\pi} \sin\frac{x}{2}\, dx = 2.$

(ii) $A = \int_0^{2\pi} \sin\frac{x}{2}\, dx$

$= 2\int_0^{\pi} \sin\frac{x}{2}\, dx$

symmetrical about $x = \pi$

$= 2(2)$ [from (i)]

$= 4$

$\therefore$ area is 4 units^2.