C.E.M. TUITION

Name : ______________________________

Review Topic : Trigonometric functions
& Applications

(Paper 2)

Year 12 - Mathematics

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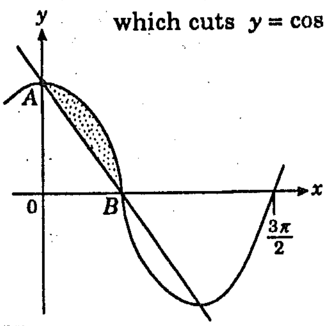
6. (a) The minute hand of a clock is 4 cm in length. What area is swept by the hand in an interval of 40 minutes? Answer in terms of π.

(b) Find the derivative of: **(i)** $\sin 2x + \cos x$ **(ii)** $\dfrac{1}{\cos x}$

(c) Find: **(i)** $\int (\sin x - \cos x)\, dx$ **(ii)** $\int \dfrac{\cos x}{\sin x + 1}\, dx$

(d) The diagram shows the graph of $y = \cos x$ and a straight line which cuts $y = \cos x$ at the points A and B.

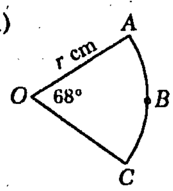


(i) Find the coordinates of the points A and B.

(ii) Show that the equation of the line passing through A and B is $y = \dfrac{\pi - 2x}{\pi}$.

(iii) Find the shaded area between $y = \cos x$ and $y = \dfrac{\pi - 2x}{\pi}$ (marked on the diagram).

7. (a)



The figure shows a sector of a circle with radius r cm. The length of the arc ABC is 7 cm.

(i) Find the value of r, to one decimal place.

(ii) Show that the area of the sector OAC is approximately 21 cm^2.

(b) Differentiate: **(i)** $\log(\sin x + \cos x)$ **(ii)** $\cos^2(3x-1)$

(c) **(i)** Sketch the graph of $y = \cos x$, where $-\pi \le x \le \pi$.

(ii) On the same number plane, graph $y = \frac{1}{2}$.

(iii) Using (i) and (ii), solve $\cos x > \frac{1}{2}$ for $-\pi \le x \le \pi$.

(d) The area bounded by the curve $y = \sec x$, the x axis and the lines $x = 0$ and $x = \frac{\pi}{3}$ is rotated about the x axis. Find the volume of the solid formed.

8. (a) If $y'' = -3\cos x - 2\sin x$ and when $x = 0$, $y' = 0$, $y = 5$, find y in terms of x.

(b) Find the equation of the tangent to the curve $y = x\cos x$ at the point $x = \pi$.

(c) Solve the equation $\cos\frac{x}{2} = \frac{1}{\sqrt{2}}$ where $0 \le x \le 2\pi$.

(d) (i) Show that if $f(x) = 2\sin 2x + 1$, then $f'(x) = 4\cos 2x$.

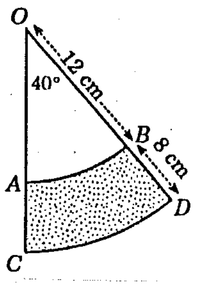
(ii) Hence, show that $\int_0^{\frac{\pi}{6}} \frac{4\cos 2x}{2\sin 2x + 1}\,dx = 1\cdot0051$, rounded off correct to four decimal places.

9. (a) (i) Show that the point $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$ lies on the curve $y = \frac{\sin x}{1 + \cos x}$.

(ii) Show that if $y = \frac{\sin x}{1 + \cos x}$, then $\frac{dy}{dx} = \frac{1}{1 + \cos x}$.

(iii) Find the equation of the tangent to the curve $y = \frac{\sin x}{1 + \cos x}$ at the point $x = \frac{\pi}{3}$.

(b)



The diagram shows AB and CD as arcs of concentric circles, with centre O. It is known that $OB = 12$ cm and $BD = 8$ cm.

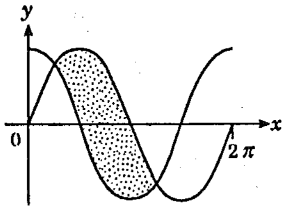
(i) Find the arc length CD, correct to 2 decimal places.

(ii) Show that the area of the shaded region is 17 868·8 cm^2.

10. (a) Show that $\frac{d}{dx}\left[x\sin x + \cos x\right] = x\cos x$, and hence evaluate

$$\int_0^{\frac{\pi}{2}} x\cos x\, dx.$$

(b)



The diagram shows the graphs of $y = \sin x$ and $y = \cos x$ for the domain $0 \le x \le 2\pi$.

(i) Show that the points of intersection of $y = \sin x$ and $y = \cos x$ are $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$.

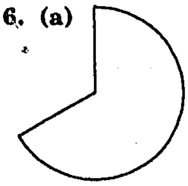
(ii) Show that $y = \sin x$ cuts the x axis at $x = 0$, π and 2π, while $y = \cos x$ cuts the x axis at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

(iii) Show that the tangent to $y = \sin x$ at $x = \pi$ is parallel to the tangent to $y = \cos x$ at $x = \frac{\pi}{2}$.

(iv) Find the exact area of the shaded region.

(c) Find the volume of the solid formed when the curve $y = \tan x$ is rotated about the x axis between $x = 0$ and $x = \frac{\pi}{4}$. Leave your answer in terms of π.

6. (a)



40 minutes

$= \frac{40 \text{ min}}{1 \text{ h}} \times 2\pi$

$= \frac{40}{60} \times 2\pi$

$= \frac{2}{3} \times 2\pi$

$= \frac{4\pi}{3}$ radians

$\therefore \quad A = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} . 4^2 . \frac{4\pi}{3}$

$= \frac{32\pi}{3}$

$\therefore$ area is $\frac{32\pi}{3}$ cm^2.

(b) (i) $\frac{d}{dx}[\sin 2x + \cos x]$

$= 2\cos 2x - \sin x.$

(ii) $\frac{d}{dx}\left[\frac{1}{\cos x}\right]$

$= \frac{d}{dx}\left[(\cos x)^{-1}\right]$

$= -1(\cos x)^{-2} . -\sin x.$

Chain Rule

$= \frac{\sin x}{\cos^2 x}$

$= \frac{\sin x}{\cos x} . \frac{1}{\cos x}$

$= \tan x . \sec x.$

(c) (i) $\int (\sin x - \cos x)\, dx$

$= -\cos x - \sin x + c$

(ii) $\int \frac{\cos x}{\sin x + 1}\, dx$

$= -\int \frac{-\cos x}{\sin x + 1}\, dx$

$= -\log_e(\sin x + 1) + c.$

(d) (i) For A:

subs. $x = 0$ in $y = \cos x$

$\therefore \quad y = \cos 0$

$\therefore \quad y = 1$

$\therefore A(0, 1).$

For B:

subs. $y = 0$ in $y = \cos x$

$0 = \cos x$

$\therefore \quad \cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$\therefore \quad B\left(\frac{\pi}{2}, 0\right)$

$\therefore A(0, 1)$ and $B\left(\frac{\pi}{2}, 0\right)$.

(ii) Eqn. of AB:

$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

$\frac{y - 1}{x - 0} = \frac{0 - 1}{\frac{\pi}{2} - 0}$

$\therefore \quad \frac{y - 1}{x} = \frac{-1}{\frac{\pi}{2}}$

$\therefore \quad \frac{y - 1}{x} = \frac{-2}{\pi}$

$\therefore \quad \pi y - \pi = -2x$

$\therefore \quad \pi y = \pi - 2x$

$\therefore \quad y = \frac{\pi - 2x}{\pi}.$

(iii) Area $= \int_0^{\frac{\pi}{2}} \cos x\, dx$

$- \int_0^{\frac{\pi}{2}} \frac{\pi - 2x}{\pi}\, dx$

$= \int_0^{\frac{\pi}{2}} \cos x\, dx$

$- \frac{1}{\pi} \int_0^{\frac{\pi}{2}} (\pi - 2x)\, dx$

$= [\sin x]_0^{\frac{\pi}{2}} - \frac{1}{\pi}\left[\pi x - x^2\right]_0^{\frac{\pi}{2}}$

$= \left(\sin \frac{\pi}{2} - \sin 0\right)$

$- \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - \frac{\pi^2}{4}\right) - (0)\right]$

$= (1 - 0) - \frac{1}{\pi}\left(\frac{\pi^2}{4}\right)$

$= 1 - \frac{\pi}{4}$

$= \frac{4 - \pi}{4}$

$\therefore$ area is $\frac{4 - \pi}{4}$ units^2.

7. (a) (i) $68° \to$ radians

$\therefore \quad 68 \times \frac{\pi}{180}$

$= 1{\cdot}186\,823\,9$

$= \theta$ (from calc.)

Now, $\ell = r\theta$

$\therefore \quad 7 = r(1{\cdot}186\,823\,9)$

$r = \frac{7}{1{\cdot}186\,823\,9}$

$= 5{\cdot}898\,095$

(from calc.)

$= 5{\cdot}90$ (2 dec. pl.)

(ii) $A = \frac{1}{2} r^2 \theta$

$= \frac{1}{2} \times 5{\cdot}9^2 \times 1{\cdot}186\,823\,9$

$= 20{\cdot}656\,67$ (from calc.)

$= 21$ (to nearest whole)

$\therefore$ area approximately 21 metres^2.

(b) (i) $\frac{d}{dx}[\log(\sin x + \cos x)]$

$= \frac{\cos x - \sin x}{\sin x + \cos x}$

$\frac{d}{dx}[\log f(x)] = \frac{f'(x)}{f(x)}$

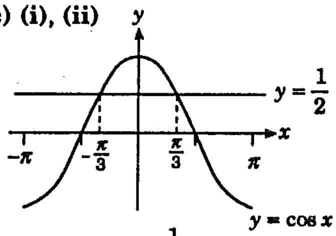
(ii) $\frac{d}{dx}\left[\cos^2(3x - 1)\right]$

$= \frac{d}{dx}\left[(\cos(3x - 1))^2\right]$

$= 2\cos(3x - 1) . -3\sin(3x - 1)$

$= -6\sin(3x - 1)\cos(3x - 1).$

(c) (i), (ii)



(iii) $\cos x > \frac{1}{2}$ means $y = \cos x$ is 'above' $y = \frac{1}{2}$. But $\cos x = \frac{1}{2}$ when $x = \frac{\pi}{3}, -\frac{\pi}{3}$,

$\therefore \quad \cos x > \frac{1}{2}$ when

$-\frac{\pi}{3} < x < \frac{\pi}{3}.$

(d) $V = \pi \int_a^b y^2\, dx$

$= \pi \int_0^{\frac{\pi}{3}} (\sec x)^2\, dx$

$= \pi \int_0^{\frac{\pi}{3}} \sec^2 x\, dx$

$= \pi[\tan x]_0^{\frac{\pi}{3}}$

$= \pi\left[\tan \frac{\pi}{3} - \tan 0\right]$

$= \pi\left[\sqrt{3} - 0\right]$

$= \sqrt{3}\pi$

$\therefore$ volume is $\sqrt{3}\pi$ units^3.

8. (a) $y'' = -3\cos x - 2\sin x$

$\therefore\ y' = \int(-3\cos x - 2\sin x)\,dx$

$\therefore\ y' = -3\sin x + 2\cos x + C$

Subs. in $x = 0,\ y' = 0$

$\therefore\ 0 = -3\sin 0 + 2\cos 0 + C$

$\therefore\ 0 = 2 + C$

$\therefore\ C = -2$

$\therefore\ y' = -3\sin x + 2\cos x - 2.$

Now, $y = \int(-3\sin x + 2\cos x - 2)\,dx$

$\therefore\ y = 3\cos x + 2\sin x - 2x + k.$

Subs. in $x = 0,\ y = 5$

$\therefore\ 5 = 3\cos 0 + 2\sin 0 - 2(0) + k$

$\therefore\ 5 = 3 + k$

$\therefore\ k = 2$

$\therefore\ y = 3\cos x + 2\sin x - 2x + 2.$

(b) $y = x\cos x$

$\therefore\ \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

[$u = x,\quad v = \cos x$]

$= \cos x . 1 + x . -\sin x$

$\therefore\ \frac{dy}{dx} = \cos x - x\sin x.$

Subs. in $x = \pi$ in $\frac{dy}{dx}$

$\therefore\ \frac{dy}{dx} = \cos\pi - \pi\sin\pi$

$= -1 - \pi(0)$

$= -1$

$\therefore$ grad. of tangent $= -1$.

Subs. in $x = \pi$ in y

$\therefore\ y = x\cos x$

$= \pi\cos\pi$

$= \pi(-1)$

$= -\pi$

$\therefore$ point $(\pi, -\pi)$, grad. $(m) = -1$

$\therefore\ y - y_1 = m(x - x_1)$

$y + \pi = -1(x - \pi)$

$y + \pi = -x + \pi$

$\therefore\ x + y = 0$

$\therefore$ eqn. of tangent is $x + y = 0$.

(c) $\cos\frac{x}{2} = \frac{1}{\sqrt{2}}$

$\therefore\ \frac{x}{2} = \frac{\pi}{4}, \frac{7\pi}{4}$

$\therefore\ x = \frac{\pi}{2}, \frac{7\pi}{2}$

(cannot have $\frac{7\pi}{2}$)

$\therefore\ x = \frac{\pi}{2}.$

(d) (i) $f(x) = 2\sin 2x + 1$

$f'(x) = 2.2\cos 2x$

$= 4\cos 2x.$

(ii) $\int_0^{\frac{\pi}{6}} \frac{4\cos 2x}{2\sin 2x + 1}\,dx$

$= \left[\log_e(2\sin 2x + 1)\right]_0^{\frac{\pi}{6}}$

$= \log_e\left(2\sin\frac{\pi}{3} + 1\right) - \log_e(2\sin 0 + 1)$

$= \log_e\left(\sqrt{3} + 1\right) - \log_e 1$

$= \log_e\left(\sqrt{3} + 1\right)$

$= 1{\cdot}005\,052\,5$ (from calculator)

$= 1{\cdot}0051$ (4 dec. pl.)

$\therefore\ \int_0^{\frac{\pi}{6}} \frac{4\cos 2x}{2\sin 2x + 1}\,dx = 1{\cdot}0051$ (4 dec. pl.).

9. (a) (i) Subs. $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$

in $y = \frac{\sin x}{1 + \cos x}$

$\therefore$ LHS $= y = \frac{\sqrt{3}}{3}$

RHS $= \frac{\sin\frac{\pi}{3}}{1 + \cos\frac{\pi}{3}}$

$= \frac{\sqrt{3}}{2} \div \left(1 + \frac{1}{2}\right)$

$= \frac{\sqrt{3}}{2} \div \frac{3}{2}$

$= \frac{\sqrt{3}}{\not{2}} \times \frac{\not{2}}{3}$

$= \frac{\sqrt{3}}{3}$

$\therefore$ LHS = RHS

$\therefore\ \left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$ lies on $y = \frac{\sin x}{1 + \cos x}$.

(ii) $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

[$u = \sin x,\quad v = 1 + \cos x$]

$= \frac{(1 + \cos x).\cos x - \sin x. -\sin x}{(1 + \cos x)^2}$

$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$

$= \frac{1 + \cos x}{(1 + \cos x)^2}$

[$\sin^2 x + \cos^2 x = 1$]

$\therefore\ \frac{dy}{dx} = \frac{1}{1 + \cos x}.$

(iii) Subs. $x = \frac{\pi}{3}$ in $\frac{dy}{dx}$

$\therefore\ \frac{dy}{dx} = \frac{1}{1 + \cos\frac{\pi}{3}}$

$= \frac{1}{1 + \frac{1}{2}}$

$= 1 \div \frac{3}{2}$

$= 1 \times \frac{2}{3}$

$= \frac{2}{3}$

$\therefore$ point $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$ and grad. $(m) = \frac{2}{3}$

$\therefore\ y - y_1 = m(x - x_1)$

$y - \frac{\sqrt{3}}{3} = \frac{2}{3}\left(x - \frac{\pi}{3}\right)$

[Mult. by 9]

$9y - 3\sqrt{3} = 6x - 2\pi$

$\therefore\ 6x - 9y - 2\pi + 3\sqrt{3} = 0.$

(b) (i) $40° \to$ radians

$\therefore\ \theta = 40 \times \frac{\pi}{180}$

$= \frac{2\pi}{9}$ radians

$= 0{\cdot}698\,131\,7$ (from calc.)

Now, $\ell = r\theta$

$= 20 \times 0{\cdot}698\,131\,7$

$= 13{\cdot}962\,634$ (from calc.)

$= 13{\cdot}96$ (two dec. pl.)

$\therefore$ arc length is $13{\cdot}96$ cm.

(ii) Let $r_1 = 20,\ r_2 = 12$

$\therefore$ Area $= \frac{1}{2}r_2^2\theta - \frac{1}{2}r_1^2\theta$

[subtract areas of sectors]

$= \frac{1}{2}\theta(r_1^2 - r_2^2)$

$= \frac{1}{2} \times \frac{2\pi}{9}(20^2 - 12^2)$

$= \frac{\pi}{9}(256)$

$= 89{\cdot}360\,858$

$= 89{\cdot}4.$

Area of shaded region is $89{\cdot}4$ cm^2.

10. (a) $\frac{d}{dx}[x\sin x+\cos x]$

$=\sin x.1+x.\cos x-\sin x$

$=\sin x+x\cos x-\sin x$

$=x\cos x$

$\therefore \frac{d}{dx}[x\sin x+\cos x]=x\cos x.$

$\therefore \int_0^{\frac{\pi}{2}} x\cos x\,dx$

$=[x\sin x+\cos x]_0^{\frac{\pi}{2}}$ (from above)

$=\left(\frac{\pi}{2}\sin\frac{\pi}{2}+\cos\frac{\pi}{2}\right)-(0\sin 0+\cos 0)$

$=\left(\frac{\pi}{2}+0\right)-(0+1)$

$=\frac{\pi}{2}-1$

$=\frac{\pi-2}{2}$

$\therefore \int x\cos x\,dx=\frac{\pi-2}{2}.$

(b) (i) $y=\sin x,\ y=\cos x$

$\therefore \sin x=\cos x$

Divide by $\cos x$:

$\tan x=1$

$\therefore x=\frac{\pi}{4},\frac{5\pi}{4},\ldots$

Subs. $x=\frac{\pi}{4}$ in $y=\sin x$

$\therefore y=\sin\frac{\pi}{4}$

$=\frac{1}{\sqrt{2}}\quad\therefore \left(\frac{\pi}{4},\frac{1}{\sqrt{2}}\right).$

Subs. $x=\frac{5\pi}{4}$ in $y=\sin x$

$\therefore y=\sin\frac{5\pi}{4}$

$=-\frac{1}{\sqrt{2}}$

$\therefore \left(\frac{5\pi}{4},-\frac{1}{\sqrt{2}}\right)$

$\therefore$ points of int. are $\left(\frac{\pi}{4},\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4},-\frac{1}{\sqrt{2}}\right).$

(ii) Cuts x axis, $\therefore y=0$

Subs. $y=0$ in $y=\sin x$

$\therefore \sin x=0$

$\therefore x=0,\pi,2\pi,\ldots$

$\therefore$ cuts x axis at $0,\pi,2\pi$.

Now, subs. $y=0$ in $y=\cos x$

$\therefore \cos x=0$

$\therefore x=\frac{\pi}{2},\frac{3\pi}{2},\ldots$

$\therefore$ cuts x axis at $\frac{\pi}{2},\frac{3\pi}{2}$.

(iii) For $y=\sin x$

$\frac{dy}{dx}=\cos x$

Subs. $x=\pi$ in $\frac{dy}{dx}$

$\therefore \frac{dy}{dx}=\cos\pi$

$\therefore \quad =-1$

$\therefore$ grad. of tangent to $y=\sin x$ at $x=\pi$ is -1.

For $y=\cos x$

$\frac{dy}{dx}=-\sin x$

Subs. $x=\frac{\pi}{2}$ in $\frac{dy}{dx}$

$\therefore \frac{dy}{dx}=-\sin\frac{\pi}{2}$

$\therefore \quad =-1$

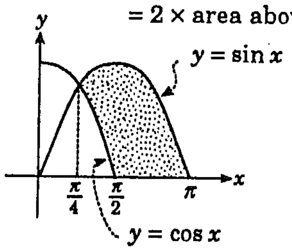
$\therefore$ grad. of tangent to $y=\cos x$ at $x=\frac{\pi}{2}$ is -1.

$\therefore$ tangents have same gradient

$\therefore$ tangents are parallel.

(iv) Shaded region above x axis = shaded region below x axis

$\therefore$ Area of shaded region = 2 × area above x axis.



$=2\times\left[\int_{\frac{\pi}{4}}^{\pi}\sin x\,dx-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos x\,dx\right]$

$=2\left[[-\cos x]_{\frac{\pi}{4}}^{\pi}-[\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right]$

$=2\left[\left(-\cos\pi+\cos\frac{\pi}{4}\right)-\left(\sin\frac{\pi}{2}-\sin\frac{\pi}{4}\right)\right]$

$=2\left[-(-1)+\frac{1}{\sqrt{2}}-\left(1-\frac{1}{\sqrt{2}}\right)\right]$

$=2\left[1+\frac{1}{\sqrt{2}}-1+\frac{1}{\sqrt{2}}\right]$

$=2\left[\frac{2}{\sqrt{2}}\right]$

$=\frac{4}{\sqrt{2}}$

$=2\sqrt{2}$

$\boxed{\frac{4}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{4\sqrt{2}}{2}=2\sqrt{2}}$

$\therefore$ area is $2\sqrt{2}$ units^2.

(c) $V=\pi\int_a^b y^2\,dx$

$=\pi\int_0^{\frac{\pi}{4}}(\tan x)^2\,dx$

$=\pi\int_0^{\frac{\pi}{4}}\tan^2 x\,dx$

$=\pi\int_0^{\frac{\pi}{4}}(\sec^2 x-1)\,dx$

$\boxed{\tan^2 x=\sec^2 x-1}$

$=\pi[\tan x-x]_0^{\frac{\pi}{4}}$

$=\pi\left[\left(\tan\frac{\pi}{4}-\frac{\pi}{4}\right)-0\right]$

$=\pi\left[1-\frac{\pi}{4}\right]$

$\therefore$ volume is $\pi\left(1-\frac{\pi}{4}\right)$ units^3.