NAME :

Centre of Excellence in Mathematics

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YEAR 12 – MATHS EXT.2

REVIEW TOPIC (PAPER 1): ELLIPSE

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(1) (b) The ellipse $\mathcal{E}$ has the equation $x^2 + \frac{y^2}{4} = 1$.

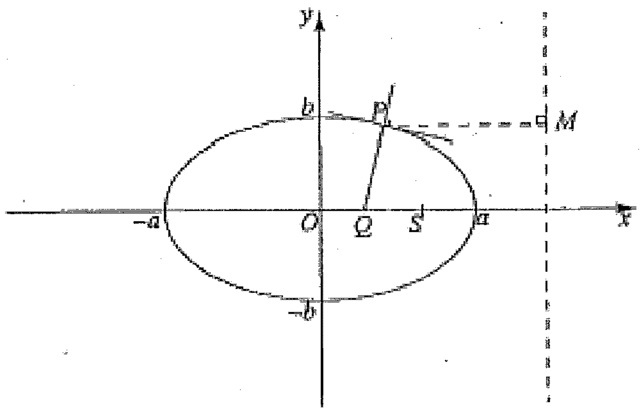
(i) Find the eccentricity and the foci of $\mathcal{E}$. **2**

(ii) Find the length of the major and minor axes of $\mathcal{E}$. **1**

(iii) Write down the equations of the directrices of $\mathcal{E}$. **1**

(iv) Sketch $\mathcal{E}$. **1**

(2) (c) 6



Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, drawn above, with eccentricity e.

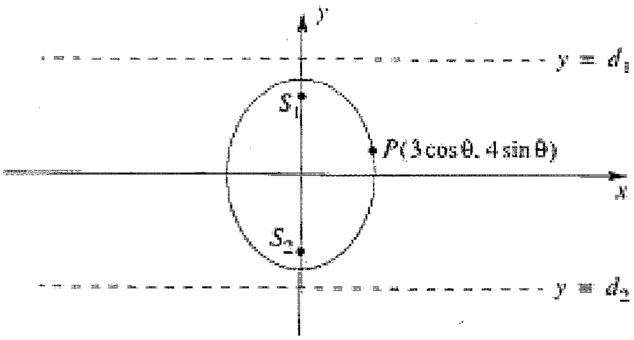
(i) Write down in terms of a and e the coordinates of the focus S, and the equation of the associated directrix.

(ii) Show that the equation of the normal to the ellipse at the point $P(x_1, y_1)$ is given by

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2.$$

(iii) Let Q be the x-intercept of the normal and let M be the foot of the perpendicular from P to the directrix as shown in the diagram. Show that $QS = e^2 PM$.

(3) (a)



The diagram above shows an ellipse with parametric equation

$$x = 3\cos\theta$$
$$y = 4\sin\theta$$

(i) Write down the cartesian equation of the ellipse. 1

(ii) Find the coordinates of the foci S_1 and S_2. 2

(iii) Find the equation of the directrices $y = d_1$ and $y = d_2$. **2**

(iv) By using a characterisation of an ellipse as a locus, or otherwise, show that $S_1P + S_2P = 8$. **2**

(4) a) i) For the ellipse $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$, state the equation of the tangents at $P(a\cos\theta, b\sin\theta)$ and at the ends of the major axes. 1

ii) Find the coordinates of the points Q and R where the tangent at P meets the two tangents at the extremities of the major axis 2

iii) Hence prove that the interval QR subtends a right angle at either focus 4

(5) (b) An ellipse has equation $\frac{x^2}{25}+\frac{y^2}{9}=1$

(i) Show that this is the equation of the locus of a point P*(x,y)* moving such that the sum of its distances from A (4, 0) and B (- 4, 0) is 10 units.

4

(ii) Calculate the eccentricity of this ellipse. 2

(iii) State the equations of the directrices 1 : this ellipse.

(iv) Find the equation of the tangent to the curve at a point Q (a, b) which lies on the ellipse. 2

SOLUTIONS

(1) (b) (i) $x^2 + \frac{y^2}{4} = 1$ so, $a = 2$ and $b = 1$

Now $b^2 = a^2(1 - e^2)$

$1 = 4(1 - e^2)$

$e = \frac{\sqrt{3}}{2}$ **(1 mark)**

The foci are located at $(0, \pm ae)$, that is at $(0, \sqrt{3})$ and $(0, -\sqrt{3})$. **(1 mark)**

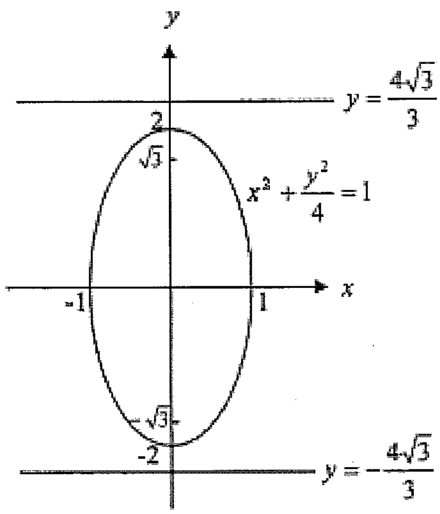
(ii) The length of the major axis is 4 units and the length of the minor axis is 2 units. **(1 mark)**

(iii) The equations of the directrices are given by $y = \pm\frac{a}{e}$

$= \pm 2 \div \frac{\sqrt{3}}{2}$

$= \pm\frac{4\sqrt{3}}{3}$ **(1 mark)**

(iv)



(1 mark)

(2) (c) (i) The focus is $S(ae, 0)$. The directrix is $x = \frac{a}{e}$. ✓

(ii) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$$

So the slope of the normal at $P(x_1, y_1)$ is $\frac{a^2 y_1}{b^2 x_1}$ ✓

The equation of the normal is $y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$

$b^2 x_1(y - y_1) = a^2 y_1(x - x_1)$ ✓

$a^2 x y_1 - b^2 x_1 y = a^2 x_1 y_1 - b^2 x_1 y_1$

$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$ ✓ (divide both sides by $x_1 y_1$)

(iii) Q is the point $\left(\frac{a^2 - b^2}{a^2} x_1, 0\right)$ or $(e^2 x_1, 0)$ (from $a^2(1 - e^2) = b^2$).

so, $QS = |e^2 x_1 - ae|$

$= e|ex_1 - a|$ ✓

Also, $PM = \frac{a}{e} - x_1$

so $e^2 PM = e^2\left(\frac{a}{e} - x_1\right)$

$= e(a - ex_1)$ ✓

Hence $QS = e^2 PM$. ✓

(3) (a) (i) $x = 3\cos\theta,\ y = 4\sin\theta$

$\frac{x}{3} = \cos\theta$

$\frac{y}{4} = \sin\theta$

Now $\cos^2\theta + \sin^2\theta = 1$

so $\frac{x^2}{9} + \frac{y^2}{16} = 1$ ✓

(ii) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

The foci are $S_1(0, ae)$ and $S_2(0, -ae)$.

Here $a = 4, b = 3$

Hence $e^2 = 1 - \left(\frac{b}{a}\right)^2$

$= 1 - \frac{9}{16}$

$= \frac{7}{16}$

$e = \frac{\sqrt{7}}{4}$ ✓

Thus $S_1 = (0, \sqrt{7})$

$S_2 = (0, -\sqrt{7})$ ✓

(iii) The upper directrix is $y = \frac{a}{e}$.

$y = \frac{4}{\frac{\sqrt{7}}{4}}$

$y = \frac{16}{\sqrt{7}}$ ✓

The lower directrix is $y = -\frac{a}{e}$.

$y = \frac{-4}{\frac{\sqrt{7}}{4}}$

$y = -\frac{16}{\sqrt{7}}$ ✓

(iv) $S_1P = ePM_1$ where PM_1 is the distance from P to the line $y = d_1$

$PM_1 = d_1 - y$

$PM_2 = y - d_2$ ✓

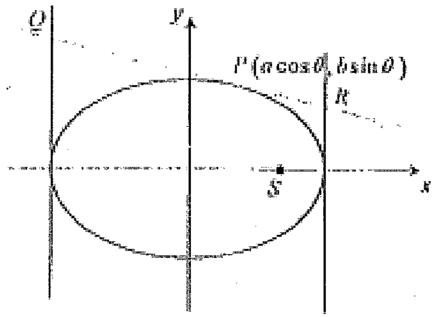
Hence $S_1P + S_2P = e(PM_1 + PM_2)$

$= (d_1 - d_2)$

$= \frac{\sqrt{7}}{4}\left(\frac{32}{\sqrt{7}}\right)$

$= 8$ ✓

(4) a)



i)

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1 \qquad 1$$

ii)

$$Q:\ x=-a \ \Rightarrow\ -\cos\theta+\frac{y\sin\theta}{b}=1$$

$$y=\frac{b(1+\cos\theta)}{\sin\theta} \qquad 1$$

$$Q\left(-a,\frac{b(1+\cos\theta)}{\sin\theta}\right)$$

$$R:\ x=a \ \Rightarrow\ \cos\theta+\frac{y\sin\theta}{b}=1$$

$$y=\frac{b(1-\cos\theta)}{\sin\theta} \qquad 1$$

$$R\left(a,\frac{b(1-\cos\theta)}{\sin\theta}\right)$$

iii) $S(ae,0)$

$$m_{SQ}=\frac{\frac{b(1+\cos\theta)}{\sin\theta}}{-a-ae}=\frac{b(1+\cos\theta)}{-a(1+e)\sin\theta} \qquad 1$$

$$m_{SQ}=\frac{\frac{b(1-\cos\theta)}{\sin\theta}}{a-ae}=\frac{b(1-\cos\theta)}{a(1-e)\sin\theta}$$

$$m_{SQ}.m_{SR}=\frac{b(1+\cos\theta)}{-a(1+e)\sin\theta}\times\frac{b(1-\cos\theta)}{a(1-e)\sin\theta}$$

$$=\frac{b^2(1-\cos^2\theta)}{a^2(1-e^2)\sin^2\theta}=-1$$

$\therefore SQ \perp SR$

(5)(b)(i) $PA + PB = 10$

$$PA = 10 - PB$$

$$\sqrt{(x-4)^2+y^2} = 10 - \sqrt{(x-4)^2+y^2}$$

$$x^2-8x+16+y^2 = 100 - 2\sqrt{(x+4)^2+y^2} + x^2+8x+16+y^2$$

$$20\sqrt{(x+4)^2+y^2} = 100 + 16x$$

$$5\sqrt{(x+4)^2+y^2} = 25 + 4x$$

$$25(x^2+8x+16+y^2) = 625+200x+16x^2$$

$$25x^2+200x+400+25y^2 = 625+200x+16x^2$$

$$9x^2+25y^2 = 225$$

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

(b)(ii) $e^2 = 1 - \frac{b^2}{a^2}$

$$= 1 - \frac{9}{25}$$

$$= \frac{16}{25}$$

$$\therefore e = \frac{4}{5}$$

(iii) Directrices $x = \pm \frac{a}{e}$

$$\therefore x = \frac{\pm 5}{4/5}$$

$x = 6\frac{1}{4}$ and $x = -6\frac{1}{4}$

(iv) $\frac{x^2}{25}+\frac{y^2}{9}=1$

By implicit diffn.

$$\frac{2x}{25}+\frac{2y}{9}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{2x}{25}\cdot\frac{9}{2y}$$

$$=-\frac{9x}{25y}$$

$\therefore$ Grad. of tangent at (a,b) is $\frac{-9a}{25b}$

$\therefore$ Eqn is $y-b=\frac{-9a}{25b}(x-a)$

$$25by-25b^2=-9ax+9a^2$$

$$9ax+25by=25b^2+9a^2$$

$\div 225$ $$\frac{ax}{25}+\frac{by}{9}=\frac{b^2}{9}+\frac{a^2}{25}$$

$$=1$$