NAME :

Centre of Excellence in Mathematics
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YEAR 12 – MATHS EXT.2

REVIEW TOPIC (PAPER 1): VOL BY SLICING (WASHER)

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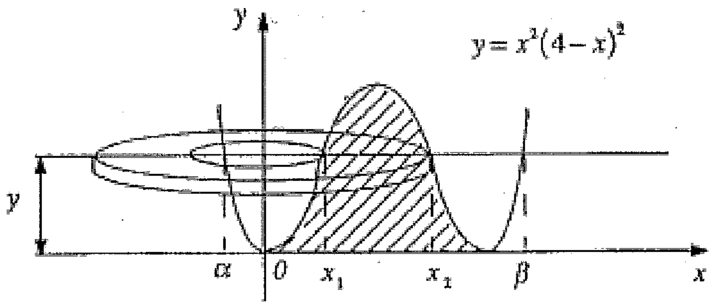
(1) (c) 9



The shaded region is rotated through one revolution about the y axis. The volume of the solid formed is found by taking slices perpendicular to the y axis. The typical slice shown in the diagram is at a height y above the x axis.

(i) Deduce that α, x_1, x_2, β, as shown in the diagram, are roots of

$x^4 - 8x^3 + 16x^2 - y = 0$.

(ii) Use the symmetry in the graph to explain why $\frac{x_1 + x_2}{2} = 2$ and $\frac{\alpha + \beta}{2} = 2$.

Hence, by considering the coefficients of the equation in (i), show that $\alpha\beta = -x_1 x_2$ and deduce that $x_1 x_2 = \sqrt{y}$ and $x_2 - x_1 = 2\sqrt{4 - \sqrt{y}}$.

(iii) Show that the volume of the solid of revolution is given by $V = 8\pi \int_0^{16} \sqrt{4-\sqrt{y}}\ dy$.

Use the substitution $y=(4-u)^2$ to evaluate this integral and find the exact volume.

(2) (a) A lifebelt mould is made by rotating the circle $x^2 + y^2 = 64$ through one complete revolution about the line $x = 28$, where all the measurements are in centimetres.

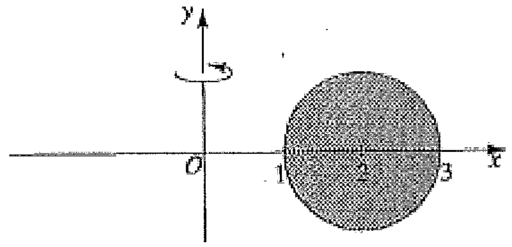
(i) Use the method of slicing to show that the volume $V\text{ cm}^3$ of the lifebelt is given by 5

$$V = 112\pi \int_{-8}^{8} \sqrt{64 - y^2}\ dy.$$

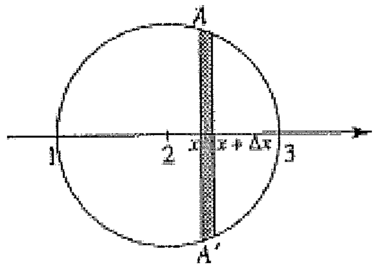
(ii) Find the exact volume of the lifebelt. 2

(3) (b) Find the volume of rotation when the region bounded by the x and y axes , $x = 2$ and the curve $y = \frac{1}{x^2 - 4x + 13}$ is rotated about the y axis .

(4) (a) 5



The circular region $(x-2)^2+y^2 \leq 1$ is rotated about the y-axis. (The resulting doughnut-shaped solid is called a *torus*.)



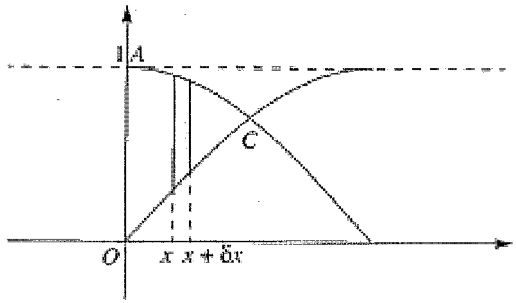
(i) Show that $AA' = 2\sqrt{1-(x-2)^2}$.

(ii) Show that the volume ΔV obtained when a typical strip of height AA' and thickness Δx is rotated about the y-axis is given by

$$\Delta V \doteq 4\pi x\sqrt{1-(x-2)^2}\ \Delta x$$

(iii) Find the total volume of the solid generated.

(5) (c) The diagram below shows part of the graphs of $y = \cos x$ and $y = \sin x$. The graph of $y = \cos x$ meets the y axis at A, and the C is the first point of intersection of the two graphs to the right of the y axis.



The region OAC is to be rotated about the line $y = 1$.

(i) Write down the coordinates of the point C. 1

(ii) The shaded strip of width δx shown in the diagram is rotated about the line $y = 1$. Show that the volume δV of the resulting slice is given by 2

$$\delta V = \pi(2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\delta x.$$

(iii) Hence evaluate the total volume when the region OAC is rotated about the line $y = 1$. 4

SOLUTIONS

(1) (c) (i) α, x_1, x_2, β satisfy $y = x^2(4-x)^2$

$y = x^2(4-x)^2 \Rightarrow x^2(x^2 - 8x + 16) - y = 0$

Hence α, x_1, x_2, β are roots of

$x^4 - 8x^3 + 16x^2 - y = 0.$

(ii) Since $x = 2$ is an axis of symmetry for the parabola and hence for $y = x^2(4-x)^2$, 2 is the midpt of the interval between x_1 and x_2, and of the interval between α and β.

$$\therefore \quad \frac{x_1 + x_2}{2} = \frac{\alpha + \beta}{2} = 2$$

Hence $x_1 + x_2 = \alpha + \beta = 4$, and

$$0 = \alpha\beta x_1 + \alpha\beta x_2 + x_1x_2\alpha + x_1x_2\beta$$
$$0 = (x_1 + x_2)\alpha\beta + (\alpha + \beta)x_1x_2$$
$$0 = 4\alpha\beta + 4x_1x_2 \Rightarrow \alpha\beta = -x_1x_2$$

Then

$$-y = \alpha\beta x_1x_2 = -(x_1x_2)^2 \Rightarrow x_1x_2 = \sqrt{y}$$

and $(x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_1x_2 = 16 - 4\sqrt{y}$

$\therefore x_2 > x_1 \Rightarrow x_2 - x_1 = \sqrt{16 - 4\sqrt{y}} = 2\sqrt{4 - \sqrt{y}}$

(iii) The slice has volume

$$\delta V = \pi\left(x_2^2 - x_1^2\right)\delta y$$
$$\delta V = \pi\left(x_2 + x_1\right)\left(x_2 - x_1\right)\delta y$$
$$= 8\pi\sqrt{4 - \sqrt{y}}\ \delta y$$

Hence

$$V = \lim_{\delta y \to 0} \sum_{y=0}^{16} 8\pi\sqrt{4 - \sqrt{y}}\ \delta y$$
$$= 8\pi\int_0^{16} \sqrt{4 - \sqrt{y}}\ dy$$

$y = (4-u)^2,\ u \le 4$

$dy = -2(4-u)\,du$

$y = 0 \Rightarrow u = 4$

$y = 16 \Rightarrow u = 0$

$4 - \sqrt{y} = 4 - (4 - u)$

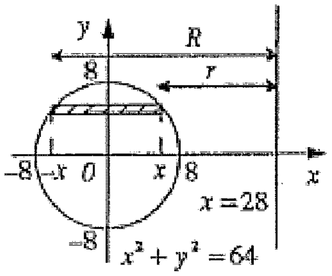
$\sqrt{4 - \sqrt{y}} = \sqrt{u}$

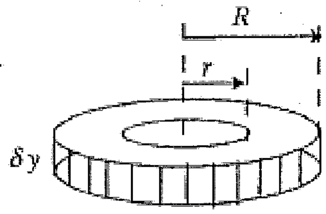
$$V = 8\pi\int_4^0 u^{\frac{1}{2}} \cdot -2(4-u)\,du$$
$$= 16\pi\int_0^4 \left(4u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) du$$
$$= 16\pi\left[\tfrac{8}{3}u^{\frac{3}{2}} - \tfrac{2}{5}u^{\frac{5}{2}}\right]_0^4$$
$$= 16\pi\left(\tfrac{64}{3} - \tfrac{64}{5}\right)$$

Hence volume is $\dfrac{2048\,\pi}{15}$ cubic units.

(2) **Answer**

(i)





$R = 28 + \sqrt{64 - y^2}$

$r = 28 - \sqrt{64 - y^2}$

Volume of slice is

$$\delta V = \pi\left(R^2 - r^2\right)\delta y$$
$$= \pi(R+r)(R-r)\,\delta y$$
$$= \pi . 56 . 2\sqrt{64 - y^2} . \delta y$$
$$V = \lim_{\delta y \to 0} \sum_{y=-8}^{8} 112\pi\sqrt{64 - y^2} . \delta y$$
$$= 112\pi \int_{-8}^{8} \sqrt{64 - y^2}\; dy$$

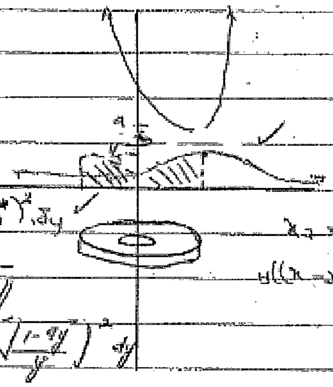
(ii) $\int_{-8}^{8} \sqrt{64 - y^2}\; dy = \frac{1}{2}\pi . 8^2 = 32\pi$ (*Area of semicircle radius* 8) $\Rightarrow V = 3584\pi^2$

Exact volume of lifebelt is $3584\pi^2$ cm^3

(3)

(b). $y = \frac{1}{x^2 - 4x + 13}$

$= \frac{1}{(x - 2)^2 + 9}$ ✓



$V = \sum^{1/13} \pi r^2 h + \sum_{1/13}^{1/9} (2 - x_1)^2 . \delta y$ ✓

$= \left[\frac{\pi r^2 4 x}{13}\right] + \int_{1/13}^{1/9} (2 -$

$\left(2 - 2 + \sqrt{\frac{1 - 9y}{y}}\right)^2 dy$

$= \frac{4\pi}{13} + \int_{1/13}^{1/9} \frac{1 - 9y}{y}\, dy$

$= \frac{4\pi}{13} + \int_{1/13}^{1/9} \frac{1}{y} - 9\; dy$

$= \frac{4\pi}{13} + \left[\ln y - 9y\right]_{1/13}^{1/9} = \frac{4\pi}{13} + \left[\left(\ln\frac{1}{9} - 1\right) - \left(\ln\frac{1}{13} - \frac{9}{13}\right)\right]$

$= \frac{4\pi}{13} + \ln\left(\frac{13}{9}\right) - \frac{4}{13}$

$x_2 = x.$

$y((x - 2)^2 + 9) = 1$

$(x - 2)^2 = \frac{1}{y} - 9$

$x - 2 = \sqrt{\frac{1}{y} - 9}$

$x = 2 \pm \sqrt{\frac{1}{y} - 9}$

$x_1 = 2 - \sqrt{\frac{1 - 9y}{y}}$

(4) (a) (i) $(x-2)^2 + y^2 = 1$

$$y^2 = 1-(x-2)^2$$

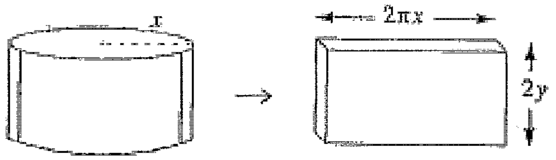
$$y = \pm\sqrt{1-(x-2)^2} \quad ✓$$

So the points are $A(x, \sqrt{1-(x-2)^2})$ and $A'(x, -\sqrt{1-(x-2)^2})$

Hence $AA' = \sqrt{1-(x-2)^2} - (-\sqrt{1-(x-2)^2})$

$$= 2\sqrt{1-(x-2)^2} \quad ✓$$

(ii)



$$\Delta V \doteqdot 2\pi x(2\sqrt{1-(x-2)^2})\Delta x$$

$$= 4\pi x\sqrt{1-(x-2)^2}\ \Delta x \quad ✓$$

(iii) $V = \int_1^3 4\pi x\sqrt{1-(x-2)^2}\ dx \quad ✓$

$$= 4\pi\int_1^3 x\sqrt{1-(x-2)^2}\ dx$$

Let $u = x-2$, $\therefore du = dx$ and $x = u+2$.

Hence $V = 4\pi\int_{-1}^{1} (u+2)\sqrt{1-u^2}\ du$

$$= 4\pi\left[\int_{-1}^{1} \underbrace{u\sqrt{1-u^2}}_{\text{odd}}\,du + \int_{-1}^{1} \underbrace{\sqrt{1-u^2}}_{\text{semicircle}}\,dx\right]$$

$$= 4\pi[0+\pi]$$

$$= 4\pi^2 \text{ cubic units} \quad ✓$$

(5) (c) (i) Put $\cos x = \sin x$

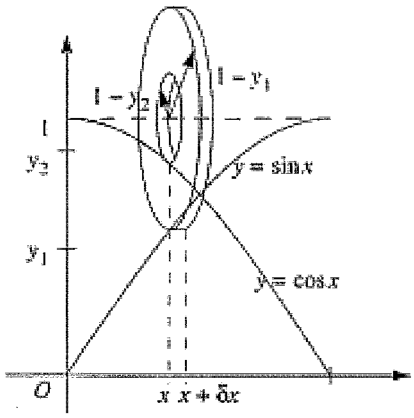
$$\Rightarrow 1 = \frac{\sin x}{\cos x}$$

Then $\tan x = 1$

$$x = \frac{\pi}{4}$$

Hence C is the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ ✓

(ii)



$$\delta V = \pi[(1-y_1)^2 - (1-y_2)^2]\delta x \quad ✓$$
$$= \pi[(1-\sin x)^2 - (1-\cos x)^2]\delta x$$
$$= \pi[1 - 2\sin x + \sin^2 x - (1 - 2\cos x + \cos^2 x)]\delta x$$
$$= \pi[2\cos x - 2\sin x + \sin^2 x - \cos^2 x]\delta x \quad ✓$$

(iii)

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{\frac{\pi}{4}} \pi[2\cos x - 2\sin x + \sin^2 x - \cos^2 x]\delta x$$
$$= \pi\int_0^{\frac{\pi}{4}} (2\cos x - 2\sin x - \cos 2x)\, dx \quad ✓$$
$$= \pi\left[2\sin x + 2\cos x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} \quad ✓$$
$$= \pi\left[2 \times \frac{1}{\sqrt{2}} + 2 \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times 1 - \left(0 + 2 - \frac{1}{2} \times 0\right)\right] \quad ✓$$
$$= \pi\left[\sqrt{2} + \sqrt{2} - \frac{1}{2} - 2\right]$$
$$= \pi\left[2\sqrt{2} - \frac{5}{2}\right] \quad ✓$$
$$= \frac{\pi}{2}[4\sqrt{2} - 5] \text{ cubic units}$$