C.E.M. TUITION

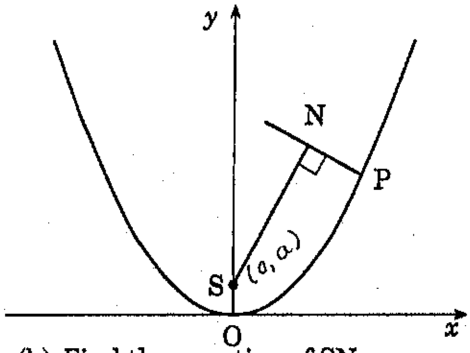
Student Name :________________________________

Review Topic : Parabola $x^2 = 4ay$

(Preliminary - Paper 1)

Year 12 - 3 Unit

1.



$P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$. SN is perpendicular to the normal at P, where S is the focus of the parabola.

(a) Show that the equation of the normal at P is

$$x + py = 2ap + ap^3$$

(b) Find the equation of SN.

(c) Find the co-ordinates (X, Y) of the point N.

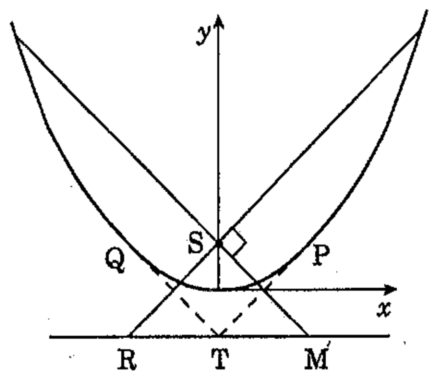
(d) Show that the locus of N is the parabola $x^2 = a(y - a)$.

2. Two perpendicular lines $x = my$ and $y = -mx$ are drawn through the origin O to meet the parabola $x^2 = 4ay$ in the points A and B.

(a) Find the co-ordinates of the points A and B.

(b) Show that the tangents at the points A and B intersect on the line $y = -4a$.

3.



Two perpendicular chords through the focus S(0, a) of the parabola $x^2 = 4ay$ meet the directrix in R and M respectively. P($2ap$, ap^2) and Q($2aq$, aq^2) are two points on the parabola such that the tangents at P and Q are parallel to the two chords.

(a) Write down the equations of the tangents at P and Q.

(b) Show that the equations of the focal chords RS and MS are $y - a = px$ and $y = qx + a$ respectively.

(c) Hence, show that $pq = -1$.

(d) Show that the coordinates of T is $(a(p+q), -a)$.

(e) Show that T is also the midpoint of MR.

1. Using the given diagram: $S(0,a)$, $P(2ap, ap^2)$

(a) The normal at P is $x + py = 2ap + ap^3$ (Theory)

(b) SN $\perp$ the normal. Gradient of SN is p. The equation of SN is

$$y - a = px$$
$$y = px + a$$

(c) Solve

$$y = px + a \quad \ldots (1)$$

and

$$x + py = 2ap + ap^3 \quad \ldots (2)$$

to find $N(X, Y)$.

Put $y = px + a$ into (2)

$$\therefore \quad x + p(px + a) = 2ap + ap^3$$
$$x(1 + p^2) = ap(1 + p^2)$$
$$x = ap$$

Then $y = px + a$

$$= a(1 + p^2)$$

$\therefore$ N is $(ap, a(1 + p^2))$

(d) To find the locus of N, eliminate the parameter p.

$X = ap, \; Y = a + ap^2$

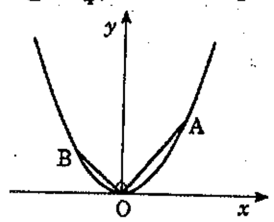
$\therefore X^2 = a^2p^2 = a \cdot ap^2$

But $ap^2 = Y - a$

$\therefore X^2 = a(Y - a)$

Hence the locus of N is a parabola $x^2 = a(y - a)$, vertex $(0, a)$, focal length $\frac{a}{4}$, concave up.

2.



(a)

Solve $x = my$ and $x^2 = 4ay$ to find the coordinates of A.

$$\therefore m^2y^2 = 4ay$$
$$y(m^2y - 4a) = 0$$

At A, $y \neq 0$, $y = \dfrac{4a}{m^2}$

$$x = m \cdot \frac{4a}{m^2} = \frac{4a}{m}$$

A is $\left(\dfrac{4a}{m}, \dfrac{4a}{m^2}\right)$

Solve $y = -mx$ and $x^2 = 4ay$ to find the coordinates of B.

B is $(-4am, 4am^2)$

(b) From $x^2 = 4ay$, $y = \dfrac{x^2}{4a}$

$$\frac{dy}{dx} = \frac{x}{2a}$$

m_1 = gradient of the tangent at A

$$m_1 = \frac{4a}{m} \cdot \frac{1}{2a} = \frac{2}{m}$$

m_2 = gradient of the tangent at B

$$m_2 = -4am \cdot \frac{1}{2a} = -2m$$

The tangents at A and B are: respectively:

$$y - \frac{4a}{m^2} = \frac{2}{m}\left(x - \frac{4a}{m}\right)$$

and

$$y - 4am^2 = -2m(x + 4am)$$

i.e. $\quad m^2y = 2mx - 4a \quad \ldots (1)$

and

$$y = -2mx - 4am^2 \quad \ldots (2)$$

Adding

$$y(1 + m^2) = -4a(1 + m^2)$$
$$y = -4a$$

Hence the tangent at A and B intersect on the line $y = -4a$.

[Students note that it is not necessary to find the x-coordinate.]

3. (a)

$$y = px - ap^2 \quad \ldots (1)$$
$$y = qx - aq^2 \quad \ldots (2)$$

(b) The gradient of line (1) is $m = p$. The focal chord parallel to (1) is $y - a = px$ or $y = px + a$. Similarly the other focal chord is $y = qx + a$.

(c) The chords are perpendicular to each other. Using $m_1m_2 = -1$, with $m_1 = p, \; m_2 = q$, we have $pq = -1$.

(d) Solving (1) and (2): Subtract (1) − (2), then:

$$0 = x(p - q) - a(p^2 - q^2)$$

$p \neq q$, divide by $p - q$.

$$x = a(p + q)$$

Then

$$y = p(ap + aq) - ap^2$$
$$= apq$$

$\therefore$ T is $(ap + aq, apq)$

But $pq = -1$

$\therefore$ T is $(ap + aq, -a)$

(e) $y = px + a$

At R, $y = -a$, so $x = \dfrac{-2a}{p}$

$\therefore$ R is $\left(\dfrac{-2a}{p}, -a\right)$

Similarly M is $\left(\dfrac{-2a}{q}, -a\right)$

The mid-point of RM is, say $N(X, Y)$, then:

$$X = \tfrac{1}{2}\left(-\frac{2a}{p} - \frac{2a}{q}\right)$$
$$= -a\left(\frac{1}{p} + \frac{1}{q}\right)$$
$$X = \frac{-a(p + q)}{pq}, \; pq = -1$$
$$X = a(p + q)$$
$$Y = \tfrac{1}{2}(-a - a) = -a$$

$\therefore$ N is $(ap + aq, -a)$

N and T have the same coordinates, hence T is the mid-point of RM.