Candidate Number

Randwick Girls' High School

2004

PRELIMINARY FINAL EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find the exact value of $\sin 15°$ in simplest surd form. **2**

(b) Given two points A (1, - 4) and B (4, 2).
Find a point P dividing them externally in the ratio 1 : 4. **2**

(c) Factorise $x^2 - 2xy + y^2 + 4x - 4y - 12$. **2**

(d) Solve the inequality: $\dfrac{2x+3}{1-x} \geq 0$ **2**

(e) Given a, b and c are three consecutive integers, show that $abc + b = b^3$. **2**

(f) When a polynomial is divided by $5x^2 + 2x - 3$, the quotient is $4x^2 - 3x - 5$ and the remainder is $x + 2$. Find the polynomial. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Given $x=\frac{1}{2}at$ and $y=at^2+2a$,

(i) find the Cartesian equation relating y and x **2**

(ii) write down the coordinates of the vertex **1**

(iii) write down the focal length. **1**

(b) The polynomial $x^3 - 3x^2 + ax + b$ has a zero ~~of multiplicity~~ 3.

~~(i) Write down the value of the zero.~~ Not in 3U Course **1**

~~(ii) Find~~ the values of a and b. **1**

(c) Solve $|x^2-5x| < 6$ **3**

(d) If $\frac{4x^2-5x-1}{(x-2)(x-1)^2}=\frac{P}{x-2}+\frac{Q}{x-1}+\frac{R}{(x-1)^2}$, find the values of P, Q and R. **3**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) From the letters of the word *FORMULAE*, how many "words" consist of 5 letters are possible if

(i) there are no restrictions **1**

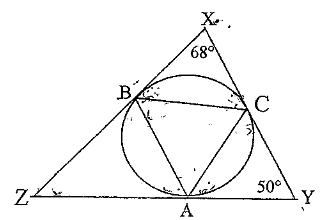
(ii) the words must start with *F* and end with *E*. **1**

Note that the "words" obtained do not necessarily have a meaning. They can be nonsense words.

(b) Differentiate with respect to *x*: **2**

$y=2x\sqrt{(2x+1)^3}$

(c) XY, YZ and ZX are tangents to the circle ABC touching it at C, A and B respectively. If $\angle X=68°$, $\angle Y=50°$, find the angles of ΔABC. **3**



(d) Find the remainder on dividing $x^{15}-1$ by x^2-1. **2**

(e) Solve the equation $\cos\theta\cos 2\theta-\sin\theta\sin 2\theta=\frac{1}{2}$ $(0°\le\theta\le 360°)$ **3**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) In a school formal, five boys and five girls are placed around each table.

Find the number of ways of seating these boys and girls around a table if

(i) there is no restriction on where a person sits around the table **1**

(ii) a particular girl wishes to sit between two particular boys **1**

(iii) two particular people do not wish to sit together **1**

(b) Use the substitution of $t = \tan\dfrac{\theta}{2}$, **3**

solve the equation $18\cos\theta + 13\sin\theta - 22 = 0$ for $0° \leq \theta \leq 360°$.

(c) In the polynomial $x^3 - 4x^2 + Ax + 4 = 0$, one root is equal to the sum of the other two roots. Find A. **3**

(d) $P(x) = x^3 - 4x^2 + kx + 6$

(i) Find the value of k if $(x + 1)$ is a factor of $P(x)$. **1**

(ii) Find the remaining factors of $P(x)$. **1**

(iii) Sketch $P(x)$. **1**

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Given the equation: $x^2 - 6x + y^2 + 8y = 0$. **2**

Draw a sketch of the geometric shape represented by the equation and clearly indicate the main features.

(b) If A (1, 3), B (-4, -2) and C (4, 2) are vertices of ΔABC, find $\angle$ACB. **3**

(c) If $\sin\theta$ and $\cos\theta$ are the two roots of the equation $3x^2 - 2x + k = 0$, find the value of k. **2**

(d) A team of 5 students are to be selected from a group of 4 girls and 7 boys to represent a school in a public-speaking competition. **2**

How many teams can be formed if there must be at least one girl in each team?

(e) Prove the identity: $\dfrac{\sin B + \sin A\cos(A+B)}{\cos B - \sin A\sin(A+B)} = \tan(A+B)$ **3**

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

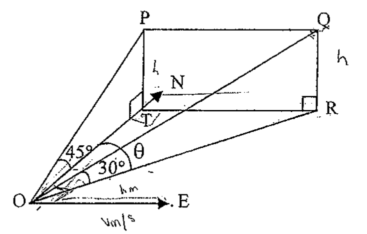
(a) (i) Sketch the parabola $x^2 = -2y$, indicating clearly the vertex (V), focus (S) and directrix (D). **1**

(ii) A and B are the points (-2, -2) and $(\frac{1}{2}, -\frac{1}{8})$ respectively. Show that AB is a focal chord of $x^2 = -2y$. **2**

(b) Solve for x: **2**

$$27^{2x-1} \div 9^{x-3} = 3^{\frac{x}{2}}$$

(c) A helicopter flies eastwards at constant height h metres with uniform speed v metres per second along the path PQ as shown. When the helicopter is at P, which is north of the point O, the angle of elevation of the helicopter from O is 45°. After t seconds, when the helicopter is at Q, the angle of elevation becomes 30°.



(i) Find the bearing θ of Q from O. **2**

(ii) If h = 1500, t = 40, find v. **2**

(d) Solve for x: **3**

$$\begin{cases} 2x - 3y - 7z = 0 \\ 5x - 2y - 8z = 0 \\ 3x^2 - 4y^2 + z^2 = 9 \end{cases}$$

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

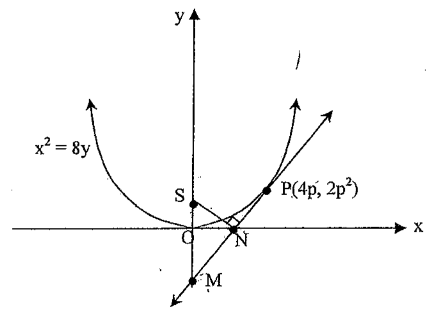
(a) Show that the line $3x - 4y + 16 = 0$ is a tangent to the circle $(x-2)^2 + (y-3)^2 = 4$. **2**

(b) Sketch the graph of the function **2**

$$f(x) = |x| + |x+2|$$

showing important features.

(c) $P(4p, 2p^2)$ is a point on the parabola $x^2 = 8y$. S is the focus. The tangent to the parabola at P meets the y-axis at M. The perpendicular from the focus S to the tangent PM meets the tangent at N.



(i) Find the co-ordinates of M and N. **4**

(ii) Find the co-ordinates of the midpoint of the interval MN. **1**

(iii) Find the equation of the locus of the midpoint of MN as P varies. **1**

(iv) A chord of contact to the parabola has equation $x - y + 10 = 0$. From what external point are the tangents drawn? **2**

End of paper

2004 Preliminary Final Mathematics Extension 1 Examination
Sample Answers

Question 1

(a) $\sin 15^\circ$

$= \sin(45^\circ - 30^\circ)$

$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}$

$= \underline{\underline{\frac{\sqrt{6}-\sqrt{2}}{4}}}$

(b) Let P be (x, y)

$x = \frac{(1)(-4)+4(1)}{1-4}$

$= 0$

$y = \frac{(-4)(-4)+(2)(1)}{1-4}$

$= -6$

$\therefore \underline{\underline{P = (0, -6)}}$

(c)

$x^2 - 2xy + y^2 + 4x - 4y - 12$

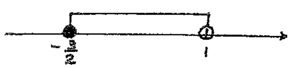
$= (x-y)^2 + 4(x-y) - 12$

$= \underline{\underline{(x-y+6)(x-y-2)}}$

(d) $\frac{2x+3}{1-x} \geq 0$

$(2x+3)(1-x) \geq 0$

$\underline{\underline{-\frac{3}{2} \leq x < 1}}$ $(x \neq 1)$



(e) Let $a = b-1$ and $c = b+1$.

$abc + b = (b-1)b(b+1) + b$

$= (b^2-1)(b) + b$

$= b^3 - b + b$

$= b^3$

$\therefore \underline{\underline{abc + b = b^3}}$

(f) Let the polynomial be $P(x)$.

$P(x) = (5x^2+2x-3)(4x^2-3x-5) + (x+2)$

$= 20x^4 - 15x^3 - 25x^2 + 8x^3 - 6x^2$

$-10x - 12x^2 + 9x + 15 + x + 2$

$= \underline{\underline{20x^4 - 7x^3 - 43x^2 + 17}}$

Question 2

(a) (i) $x = \frac{1}{2}at \Rightarrow t = \frac{2x}{a}$

Sub $t = \frac{2x}{a}$ into y

$y = a\left(\frac{2x}{a}\right)^2 + 2a$

$y = \frac{4}{a}x^2 + 2a$

$\underline{\underline{x^2 = \frac{a}{4}(y - 2a)}}$

(ii) $\underline{\underline{\text{Vertex} = (0, 2a)}}$

(iii) $\underline{\underline{\text{Focal length} = \frac{a}{16}}}$

(b) (i) Let $P(x) = x^3 - 3x^2 + ax + b$

$P'(x) = 3x^2 - 6x + a$

$P''(x) = 6x - 6$

When $6x - 6 = 0$, $x = 1$

$\therefore$ the value of the zero is 1.

(ii) $x^3 - 3x^2 + ax + b = (x-1)^3$

$= x^3 - 3x^2 + 3x - 1$

$\therefore \underline{\underline{a = 3 \text{ and } b = -1}}$

(c) $|x^2 - 5x| < 6$

$x^2 - 5x < 6$ and $-(x^2-5x) < 6$

$x^2 - 5x - 6 < 0$ and $x^2 - 5x + 6 > 0$

$(x-6)(x+1) < 0$ and $(x-2)(x-3) > 0$

$-1 < x < 6$ and $(x > 3$ OR $x < 2)$

$\therefore \underline{\underline{-1 < x < 2 \text{ OR } 3 < x < 6}}$

(d) $\frac{4x^2-5x-1}{(x-2)(x-1)^2}$

$= \frac{P(x-1)^2 + Q(x-1)(x-2) + R(x-2)}{(x-2)(x-1)^2}$

$4x^2 - 5x - 1 = P(x-1)^2 + Q(x-1)(x-2) + R(x-2)$

When $x = 1$, $4 - 5 - 1 = -R$

$\therefore \underline{\underline{R = 2}}$

When $x = 2$, $5 = P$ $\therefore \underline{\underline{P = 5}}$

$\therefore 4x^2 - 5x - 1 = 5(x-1)^2 + Q(x-1)(x-2) + 2(x-2)$

When $x = 0$, $-1 = 5 + 2Q - 4$

$\therefore \underline{\underline{Q = -1}}$

Question 3

(a) (i) ${}^8P_5 = \frac{8!}{3!} = \underline{\underline{6720}}$

(ii) ${}^6P_3 = \frac{6!}{3!} = \underline{\underline{120}}$

(b) $y = 2x(2x+1)^{\frac{3}{2}}$

$\frac{dy}{dx} = 2x \cdot \frac{3}{2}(2x+1)^{\frac{1}{2}} \cdot 2 + 2(2x+1)^{\frac{3}{2}}$

$= 6x\sqrt{2x+1} + 2\sqrt{(2x+1)^3}$

$= 2\sqrt{2x+1}\,[3x + (2x+1)]$

$= \underline{\underline{2\sqrt{2x+1}\,(5x+1)}}$

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Sample Answers

Question 3 (cont'd)

(c) $\angle XBC = \angle XCB = \angle A$

($\angle$ in alt segment)

and $2\angle A + 68^\circ = 180^\circ$

($\angle$ sum of Δ)

$\therefore 2\angle A = 180^\circ - 68^\circ \Rightarrow \underline{\underline{\angle A = 56^\circ}}$

Similarly, $\angle YCA = \angle YAC = \angle B$

($\angle$ in alt. segment)

and $2\angle B + 50^\circ = 180^\circ$

($\angle$ sum of Δ)

$\therefore 2\angle B = 180^\circ - 50^\circ \Rightarrow \underline{\underline{\angle B = 65^\circ}}$

Hence $\angle C = 180^\circ - 56^\circ - 65^\circ$

($\angle$ sum of Δ)

$\therefore \underline{\underline{\angle C = 59^\circ}}$

(d) Let $x^{15} - 1 = (x^2-1)Qx + Ax + Bx$

$x^{15} - 1 = (x-1)(x+1)Q(x) + Ax + B$

When $x = 1$, $0 = A + B$ — ①

When $x = -1$, $-2 = -A + B$ — ②

① + ② $-2 = 2B$

$\therefore B = -1 \quad A = 1$

The remainder is $\underline{\underline{x - 1}}$

(e) $\cos\theta\cos 2\theta - \sin\theta\sin 2\theta = \frac{1}{2}$

$\cos(\theta + 2\theta) = \frac{1}{2}$

$\cos 3\theta = \frac{1}{2}$

$3\theta = 60^\circ, 300^\circ, 420^\circ, 660^\circ, 780^\circ, 1020^\circ$

$\therefore \theta = 20^\circ, 100^\circ, 140^\circ, 220^\circ, 160^\circ, 340^\circ$

Question 4

(a) (i) $9! = 362880$

(ii) $2(7!) = 10080$

(iii) $7(8!) = \underline{282240}$

(b) $18\left(\frac{1-t^2}{1+t^2}\right) + 13\left(\frac{2t}{1+t^2}\right) - 22 = 0$

$18(1-t^2) + 13(2t) - 22(1+t^2) = 0$

$-40t^2 + 26t - 4 = 0$

$20t^2 - 13t + 2 = 0$

$(5t-2)(4t-1) = 0$

$\therefore t = 0.4$ OR $t = 0.25$

$\tan\frac{\theta}{2} = 0.4 \qquad \tan\frac{\theta}{2} = 0.25$

$\frac{\theta}{2} = 21^\circ 48' \qquad \frac{\theta}{2} = 14^\circ 2'$

$\theta = 43^\circ 36' \qquad \theta = 28^\circ 4'$

$\therefore \underline{\underline{\theta = 28^\circ 4' \text{ OR } 43^\circ 36'}}$

(c) Let $\alpha+\beta, \alpha, \beta$ be the 3 roots.

Sum of roots: $(\alpha+\beta) + \alpha + \beta = 4$

$\therefore \alpha + \beta = 2$

Product of roots: $(\alpha+\beta)\alpha\beta = -4$

$2\alpha\beta = -4$

$\therefore \alpha\beta = -2$

$A = (\alpha+\beta)\alpha + \alpha\beta + \beta(\alpha+\beta)$

$= 2\alpha + (-2) + \beta(2)$

$= 2(\alpha+\beta) - 2$

$= 2 \times 2 - 2$

$= \underline{\underline{2}}$

(d) (i) $P(-1) = -1 - 4 - k + 6 = 0$

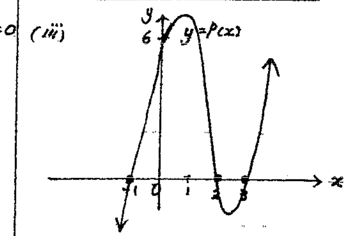
$\therefore \underline{\underline{k = 1}}$

(ii) By division,

$P(x) = (x+1)(x^2 - 5x + 6)$

$\therefore P(x) = \underline{\underline{(x+1)(x-2)(x-3)}}$

(iii)

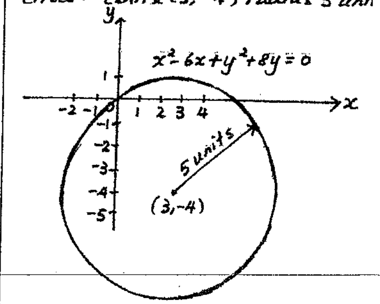


Question 5

(a) $x^2 - 6x + y^2 + 8y = 0$

$(x-3)^2 + (y+4)^2 = 5^2$

Circle: centre $(3, -4)$ radius 5 units



(b) $m_{AC} = \frac{3-2}{1-4} = -\frac{1}{3}$

$m_{BC} = \frac{2--2}{4--4} = \frac{1}{2}$

Let $\angle ACB = \theta$

$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

$= \left|\frac{\frac{1}{2} - -\frac{1}{3}}{1 + (\frac{1}{2})(-\frac{1}{3})}\right|$

$\therefore \theta = 45^\circ$

$\therefore \angle ACB = \underline{\underline{45^\circ}}$

<u>Question 5 (cont'd)</u>

(c) $\sin\theta + \cos\theta = \frac{2}{3}$

(sum of roots)

$\frac{k}{3} = \sin\theta\cos\theta$

(product of roots)

$(\sin\theta + \cos\theta)^2 = \left(\frac{2}{3}\right)^2$

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = \frac{4}{9}$

$\therefore \sin\theta\cos\theta = \frac{1}{2}\left(\frac{4}{9} - 1\right)$

$= -\frac{5}{18}$

$\therefore k = 3\sin\theta\cos\theta$

$= -\frac{5}{18} \times 3$

$\underline{\underline{k = -\frac{5}{6}}}$

(d) ${}^{11}C_5 - {}^{7}C_5 = \underline{\underline{441}}$

(e) L.H.S.

$= \dfrac{\sin B + \sin A(\cos A\cos B - \sin A\sin B)}{\cos B - \sin A(\sin A\cos B + \cos A\sin B)}$

$= \dfrac{\sin B + \sin A\cos A\cos B - \sin^2 A\sin B}{\cos B - \sin^2 A\cos B - \sin A\cos A\sin B}$

$= \dfrac{\sin B(1 - \sin^2 A) + \sin A\cos A\cos B}{\cos B(1 - \sin^2 A) - \sin A\cos A\sin B}$

$= \dfrac{\sin B\cos^2 A + \sin A\cos A\cos B}{\cos B\cos^2 A - \sin A\cos A\sin B}$

$= \dfrac{\cancel{\cos A}(\sin B\cos A + \sin A\cos B)}{\cancel{\cos A}(\cos B\cos A - \sin A\sin B)}$

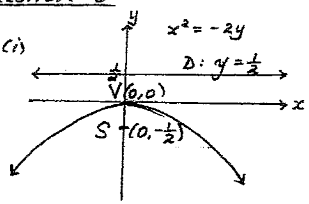
$= \dfrac{\sin(A+B)}{\cos(A+B)}$

$= \tan(A+B)$

$= RHS.$

<u>Question 6</u>

(a) (i)



(ii) Equation of AB:

$(y+2) = \left(\dfrac{-2 + \frac{1}{8}}{-2 - \frac{1}{2}}\right)(x + 2)$

$y + 2 = \frac{3}{4}(x+2)$

$4y + 8 = 3x + 6$

$3x - 4y - 2 = 0$

Sub $S(0, -\frac{1}{2})$ in eqn. of AB:

$L.H.S. = 3(0) - 4(-\frac{1}{2}) - 2$

$= 0 = RHS.$

$\therefore$ AB passes thru the focus

$\therefore$ AB is a focal chord of $x^2 = -2y$.

(b) $27^{2x-1} \div 9^{x-3} = 3^{\frac{x}{2}}$

$3^{6x-3} \div 3^{2x-6} = 3^{\frac{x}{2}}$

$(6x-3) - (2x-6) = \frac{x}{2}$

$4x + 3 = \frac{x}{2}$

$8x + 6 = x$

$7x = -6$

$\therefore x = \underline{\underline{-\frac{6}{7}}}$

(c) (i)

$\tan 45° = \frac{PT}{OT}$

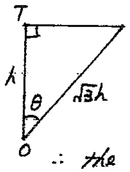
$1 = \frac{h}{OT}$

$\therefore OT = h$

$\tan 30° = \frac{QR}{OR}$

$\frac{1}{\sqrt{3}} = \frac{h}{OR}$

$OR = \sqrt{3}\,h$



$\cos\theta = \frac{h}{\sqrt{3}h}$

$\therefore \theta = 54°44'$

$\therefore$ the bearing is N54°44'E (OR 055°T)

(c) (ii) $\tan\theta = \frac{TR}{h}$

$\tan 54°44' = \frac{TR}{1500}$

$\therefore TR = 2121.14\ldots$

$v = \frac{\text{distance}}{\text{time}}$

$= \frac{2121.14\ldots}{40}$

$= 53.0$ (to 3 s.f.)

(d) $2x + 3y - 7z = 0$ —— ①

$5x - 2y - 8z = 0$ —— ②

$3x^2 - 4y^2 + z^2 = 9$ —— ③

From ① & ②

$10x + 15y - 35z = 0$
$10x - 4y - 16z = 0$
$19y - 19z = 0$
$\therefore y = z$

$4x + 6y - 14z = 0$
$15x - 6y - 24z = 0$
$19x - 38z = 0$
$\therefore x = 2z$

Sub $y = z$ and $x = 2z$ into ③

$3(2z)^2 - 4z^2 + z^2 = 9$

$12z^2 - 4z^2 + z^2 = 9$

$9z^2 = 9$

$\therefore z = \pm 1$

When $z = 1$, $y = 1$, $x = 2$

When $z = -1$, $y = -1$, $x = -2$

$\therefore \begin{cases} x = 2 \\ y = 1 \\ z = 1 \end{cases}$ OR $\begin{cases} x = -2 \\ y = -1 \\ z = -1 \end{cases}$

<u>Question 7</u>

(a) distance from (2,3) to $3x - 4y + 16 = 0$:

$\left|\dfrac{3(2) - 4(3) + 16}{\sqrt{(-4)^2 + 3^2}}\right| = \frac{10}{5} = 2$

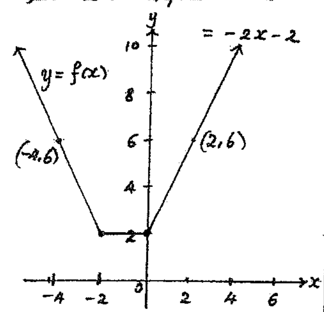
Since the perp. distance from the centre of the circle to the line equals the length of the radius the line $3x - 4y + 16 = 0$ must be a tangent.

(b) $f(x) = |x| + |x+2|$

For $x \geq 0$, $f(x) = 2x + 2$

For $-2 < x < 0$, $f(x) = -x + x + 2$
$= 2$

For $x \leq -2$, $f(x) = -x - x - 2$
$= -2x - 2$



(c)(i) $y = \frac{1}{8}x^2$

$\frac{dy}{dx} = \frac{1}{4}x$

At $x = 4p$, $\frac{dy}{dx} = p$

Equation of tangent:

$y - 2p^2 = p(x - 4p)$

$y = px - 2p^2$

When $x = 0$, $y = -2p^2$

$\therefore \underline{\underline{M = (0, -2p^2)}}$

$S = (0, 2)$

$m_{SN} = -\frac{1}{p}$ ($\perp$ to PM)

Equation of SN:

$y - 2 = -\frac{1}{p}(x - 0)$

$y = -\frac{1}{p}x + 2$

To find N:

Solve $\begin{cases} y = px - 2p^2 & \text{①} \\ y = -\frac{1}{p}x + 2 & \text{②} \end{cases}$ simultaneously

$px - 2p^2 = -\frac{1}{p}x + 2$

$\left(\frac{p^2+1}{p}\right)x = 2 + 2p^2$

$\therefore x = \dfrac{2p\cancel{(1+p^2)}}{\cancel{1+p^2}}$

$x = 2p$

Sub $x = 2p$ into ②

$y = -\frac{1}{p} \times 2p + 2$

$= 0$

$\therefore \underline{\underline{N = (2p, 0)}}$

(ii) Midpoint of $\overline{MN}$:

$\left(\frac{0 + 2p}{2}, \frac{-2p^2 + 0}{2}\right)$

$= \underline{(p, -p^2)}$

(iii) Equation of locus:

$x = p$

$y = -p^2$

$\therefore \underline{\underline{y = -x^2}}$

(iv) Let the external point be (x_0, y_0).

Equation of chord of contact is:

$xx_0 = 2a(y + y_0)$

$a = 2$ (focal length)

$\therefore xx_0 = 4(y + y_0)$

$\frac{x_0}{4}x - y - y_0 = 0$

Comparing to $x - y + 10 = 0$

$\frac{x_0}{4} = 1$ $\quad -y_0 = 10$

$\therefore x_0 = 4$ $\quad \therefore y_0 = -10$

$\therefore$ the external point is $(4, -10)$.