RANDWICK GIRLS' HIGH SCHOOL

YEAR 12 - 2008
MATHEMATICS EXTENSION 1

Assessment Task 2

Time Allowed: 1 hour

Examiner: D. Posener

General Instructions:

- Attempt all questions
- Write using black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Begin each question on a new page.

NAME: ______________________________

QUESTION	MARK
1	/16
2	/18
3	/12
4	/12
TOTAL	/58

Question 1:

(i) Evaluate $\lim_{x \to 0} \frac{5\sin x}{3x}$ (1)

(ii) Use the table of standard integrals to find $\int \sec 2x \tan 2x \, dx$ (1)

(iii) The area of the sector of a circle is $8\pi \, cm^2$ and the length of the arc bounded by this sector is $\frac{\pi}{4} cm$. Find the radius of the circle and the angle that is subtended at the centre. (2)

(iv) Evaluate $\int_0^1 \sqrt{1-x^2} \, dx$ using the substitution $x = \sin\sigma$ (4)

(v) At any point on the curve $y = f(x)$, the gradient function is given by $\frac{dy}{dx} = 2\sin^2 2x + 1$. If $y = 1$ when $x = \pi$, find the value of y when $x = 2\pi$. (4)

(vi) For the curve $y = 1 + 2\sin x - 2\sin^2 x$ show $\frac{dy}{dx} = 2\cos x(1 - 2\sin x)$.
Hence find the stationary points in the interval $0 \le x \le \frac{\pi}{2}$. Sketch the curve in this interval. (4)

Question 2:

(i) Consider the function $f(x)=3\cos^{-1}\left(\frac{x}{2}\right)$

(a) Evaluate $f(2)$ (1)

(b) State the domain and range of $y=f(x)$ (2)

(c) Draw the graph of $y=f(x)$ (2)

(ii) Differentiate $y=2\cos^{-1}\frac{x}{\sqrt{2}}-\sin^{-1}(1-x^2)$ (4)

(iii) Find the area of the region in the first quadrant bounded by the curve $y=4\cos^{-1}\left(\frac{x}{3}\right)$ and the coordinate axes. (3)

(iv) (a) Let $\tan\alpha=x$ and $\tan\beta=y$. (6)
Prove using the expansion for $\tan(\alpha+\beta)$ that
$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(b) Hence write an expression for $2\tan^{-1}x$

(c) Hence show that $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\frac{\pi}{4}$

Question 3:

(i) Evaluate $\log_5 7$ to 3 significant figures. (1)

(ii) Find $\int\frac{e^x\,dx}{\sqrt{49-e^{2x}}}$ using $u=e^x$ (2)

(iii) Find $\int\frac{4x}{1-5x^2}\,dx$ (2)

(iv) Find $\int 3xe^{6x^2+1}\,dx$ (2)

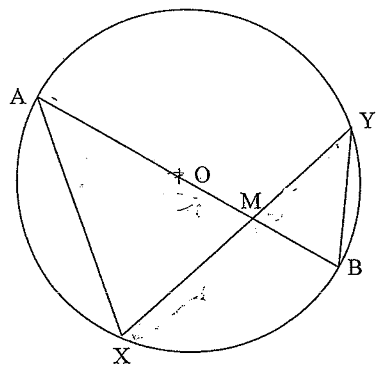
(v) (a) Differentiate $y=\ln(1+\tan x)$ (2)

(b) Hence evaluate $\int_0^{\frac{\pi}{4}}\frac{\sec^2 x}{1+\tan x}\,dx$

(vi) Find the volume of the solid of revolution formed when the area between the curve $y=\ln x$ and the x-axis and the line $x=4$ is rotated about the y-axis. (3)

Question 4:

(i)

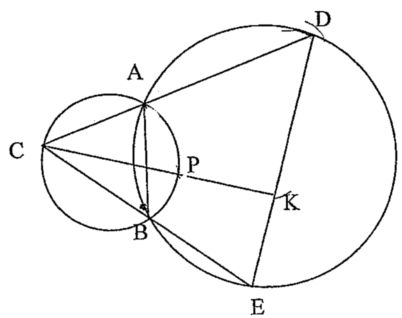


In the diagram above AB is a diameter of a circle, whose centre is O.
The chord XY passes through M, the midpoint of OB. AX and BY are joined.

(a) Prove Δ AXM ||| Δ MYB (4)

(b) If XM = 8cm and YM = 6cm, find the length of the radius of the circle. (4)

(ii)



In the diagram, the 2 circles intersect at A and B, and CAD, CBE, CPK and DKE are straight lines.

(a) Prove ∠ APC = ∠ ABC (1)

(b) Show that ADKP is a cyclic quadrilateral. (3)

Solutions

2P Q1 Q3
TM Q2 Q4.

Q1. (i) $\lim_{x\to 0} \frac{5\sin x}{3x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{5}{3}$

$= 1 \times \frac{5}{3}$

$= \frac{5}{3}$

(ii) $\int \sec 2x \tan 2x \, dx = \frac{\sec 2x}{2} + c$

(iii) $A = \frac{1}{2} r^2 \theta = 8\pi$ — (1)

$l = r\theta = \frac{\pi}{4}$ — (2)

(1) ÷ (2): $\frac{\frac{1}{2} r^2 \theta}{r\theta} = \frac{8\pi}{\frac{\pi}{4}}$

$\frac{1}{2} r = 32$

$r = 64$

Sub. into (1) $64\theta = \frac{\pi}{4}$

$\theta = \frac{\pi}{4} \times \frac{1}{64}$

$\theta = \frac{\pi}{256}$

(iv) $\int_0^1 \sqrt{1-x^2}\, dx$ Let $x = \sin\theta$

$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta} \cdot \cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta} \cdot \cos\theta \, d\theta$ ✓ when $x = 1$, $\theta = \frac{\pi}{2}$; when $x = 0$, $\theta = 0$.

$= \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$ ✓

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \cos 2\theta \, d\theta$

$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$ ✓

$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - \left(0 + \cancel{\frac{\sin 0}{2}}\right)\right]$

$= \frac{1}{2}\left[\frac{\pi}{2} + 0\right] = \frac{\pi}{4}$. ✓

(v) $\frac{dy}{dx} = 2\sin^2 2x + 1$

$y = \int 2\sin^2 2x + 1 \, dx$

$= \int 2\left(\frac{1}{2}(1 - \cos 4x)\right) + 1 \, dx$ ✓

$= \int 1 - \cos 4x + 1 \, dx$

$= \int 2 - \cos 4x \, dx$

$y = 2x - \frac{\sin 4x}{4} + c$ ✓

$x = \pi$, $y = 1$: $1 = 2\pi - \frac{\sin 4\pi}{4} + c$

$1 = 2\pi - 0 + c$

$c = 1 - 2\pi$

$\therefore y = 2x - \frac{\sin 4x}{4} + 1 - 2\pi$

when $x = 2\pi$

$y = 4\pi - \frac{\sin 8\pi}{4} + 1 - 2\pi$

$y = 4\pi + 1 - 2\pi$

$y = 2\pi + 1$. ✓

4) (vi) $y = 1 + 2\sin x - 2\sin^2 x$

$\frac{dy}{dx} = 0 + 2\cos x - 4\sin x \cdot \cos x$

$= 2\cos x - 4\sin x \cos x$

$= 2\cos x(1 - 2\sin x)$ ✓

Stat. pts occur when $\frac{dy}{dx} = 0$

$2\cos x(1 - 2\sin x) = 0$

$\cos x = 0$ or $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}$ ✓ or $x = \frac{\pi}{6}$.

when $x = \frac{\pi}{2}$

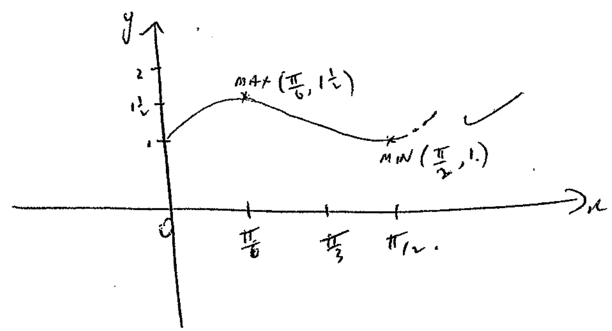
x	$\frac{\pi}{2}^-$	$\frac{\pi}{2}$	$\frac{\pi}{2}^+$
$f'(x)$	<0	0	>0

$\therefore$ min at $\left(\frac{\pi}{2}, 1\right)$

when $x = \frac{\pi}{6}$

x	$\frac{\pi}{6}^-$	$\frac{\pi}{6}$	$\frac{\pi}{6}^+$
$f'(x)$	>0	0	<0

$\therefore$ max at $\left(\frac{\pi}{6}, 1\frac{1}{2}\right)$ ✓



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Question 2:

(i) $f(x) = 3\cos^{-1}\left(\frac{x}{2}\right)$

(a) $f(2) = 3\cos^{-1} 1$

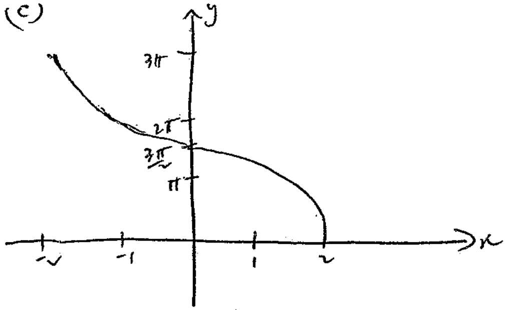
$= 3 \times 0$

$= 0$

(b) domain: $-2 \le x \le 2$

range: $0 \le y \le 3\pi$

(c)



(ii) $y = 2\cos^{-1}\frac{x}{\sqrt{2}} - \sin^{-1}(1-x^2)$

$$\frac{dy}{dx} = 2 \cdot \frac{-1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}} \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{1-(1-x^2)^2}} \times -2x$$

$$= \frac{-2}{\sqrt{2}\cdot\sqrt{1-\frac{x^2}{2}}} + \frac{2x}{\sqrt{1-(1-2x^2+x^4)}}$$

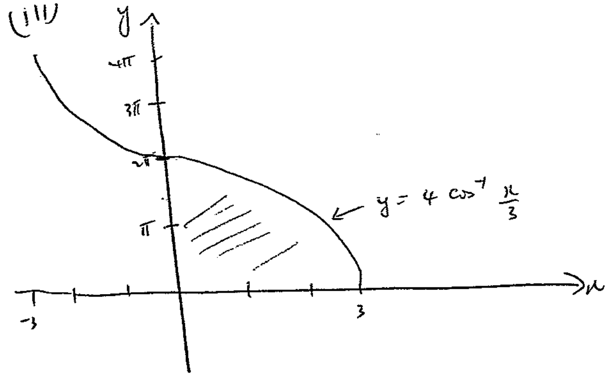
$$= \frac{-2}{\sqrt{2}\cdot\sqrt{\frac{2-x^2}{2}}} + \frac{2x}{\sqrt{2x^2-x^4}}$$

$$= \frac{-2}{\sqrt{2-x^2}} + \frac{2x}{x\sqrt{2-x^2}}$$

$$= \frac{-2}{\sqrt{2-x^2}} + \frac{2}{\sqrt{2-x^2}}$$

$= 0.$

(iii)



$y = 4\cos^{-1}\frac{x}{3}$

$\frac{y}{4} = \cos^{-1}\frac{x}{3}$

$\frac{x}{3} = \cos\frac{y}{4}$

$x = 3\cos\frac{y}{4}$

$$A = \int_0^{2\pi} 3\cos\frac{y}{4}\,dy$$

$$= 3\int_0^{2\pi} \cos\frac{y}{4}\,dy$$

$$= 3\left[\frac{\sin\frac{y}{4}}{\frac{1}{4}}\right]_0^{2\pi}$$

$$= 12\left[\sin\frac{y}{4}\right]_0^{2\pi}$$

$$= 12\left[\sin\frac{2\pi}{4} - \sin 0\right]$$

$$= 12\left[\sin\frac{\pi}{2} - 0\right]$$

$= 12(1)$

$= 12.$

(3) (iv) $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9-2x^2}} = \int_0^{3/2} \frac{1}{\sqrt{2\left(\frac{9}{2}-x^2\right)}}\,dx$

$$= \frac{1}{\sqrt2}\int_0^{3/2} \frac{1}{\sqrt{\left(\frac{3}{\sqrt2}\right)^2-x^2}}\,dx$$

$$= \frac{1}{\sqrt2}\left[\sin^{-1}\frac{x}{\frac{3}{\sqrt2}}\right]_0^{3/2}$$

$$= \frac{1}{\sqrt2}\left[\sin^{-1}\frac{\sqrt2}{3}x\right]_0^{3/2}$$

$$= \frac{1}{\sqrt2}\left[\sin^{-1}\frac{\sqrt2}{3}\cdot\frac{3}{2} - \sin^{-1}0\right]$$

$$= \frac{1}{\sqrt2}\left[\sin^{-1}\frac{\sqrt2}{2} - \sin^{-1}0\right]$$

$$= \frac{1}{\sqrt2}\left[\sin^{-1}\frac{1}{\sqrt2} - 0\right]$$

$$= \frac{1}{\sqrt2}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4\sqrt2}.$$

(v) (a) $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\cdot\tan\beta}$

$\tan\alpha = x$ $\qquad$ $\tan\beta = y$

$\therefore \alpha = \tan^{-1}x$ $\qquad$ $\therefore \beta = \tan^{-1}y$.

$\tan(\alpha+\beta) = \dfrac{x+y}{1-xy}$ (1)

$\therefore \alpha+\beta = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

$\therefore \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ (1)

(b) $2\tan^{-1}x = \tan^{-1}x + \tan^{-1}x$

(1) $= \tan^{-1}\left(\dfrac{x+x}{1-x^2}\right)$

$= \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

(c) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\dfrac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{2}{3}\div\frac{8}{9}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$ (1)

$= \tan^{-1}\left(\dfrac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)$

$= \tan^{-1}\left(\dfrac{\frac{25}{28}}{\frac{25}{28}}\right)$

$= \tan^{-1}(1)$ (1)

$= \dfrac{\pi}{4}$ (1)

Question 3:

1) (i) $\log_5 7 = \frac{\ln 7}{\ln 5}$

$= 1.21$

1) (ii) $\int \frac{e^x}{\sqrt{49 - e^{2x}}}\,dx$

use $u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x\,dx$

$= \int \frac{du}{\sqrt{49 - u^2}}$

$= \int \frac{1}{\sqrt{7^2 - u^2}}\,du$

$= \sin^{-1}\frac{u}{7} + c$

$= \sin^{-1}\frac{e^x}{7} + c.$

(2) (iii) $\int \frac{4x}{1-5x^2}\,dx$

$= \frac{-4}{10}\int \frac{-10x}{1-5x^2}\,dx$

$= -\frac{2}{5}\ln(1-5x^2) + c.$

(2) (iv) $\int 3x e^{6x^2+1}\,dx = \frac{1}{4}\int e^{6x^2+1} \cdot 12x\,dx$

$= \frac{1}{4} e^{6x^2+1} + c$

1) (v) (a) $y = \ln(1 + \tan x)$

$\frac{dy}{dx} = \frac{1}{1+\tan x} \cdot \sec^2 x$

$= \frac{\sec^2 x}{1+\tan x}$

(1) (b) $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x}\,dx$

$= \left[\ln(1+\tan x)\right]_0^{\frac{\pi}{4}}$

$= \ln(1 + \tan\frac{\pi}{4}) - \ln(1+0)$

$= \ln(1+1) - \cancel{\ln 1}^{\,0}$

$= \ln 2.$

removed questions

:) (vi) (a) $\frac{d}{dx}\left(x^2 - \ln(x^2+1)\right)$

$= 2x - \frac{1}{x^2+1} \cdot 2x$

$= 2x - \frac{2x}{x^2+1}$

$= \frac{2x(x^2+1) - 2x}{x^2+1}$

$= \frac{2x^3 + 2x - 2x}{x^2+1}$

$= \frac{2x^3}{x^2+1}.$

(b) $\int_{-1}^{1} \frac{2x^3}{x^2+1}\,dx = \left[x^2 - \ln(x^2+1)\right]_{-1}^{1}$

$\therefore \int_{-1}^{1} \frac{x^3}{x^2+1}\,dx = \frac{1}{2}\left[x^2 - \ln(x^2+1)\right]_{-1}^{1}$

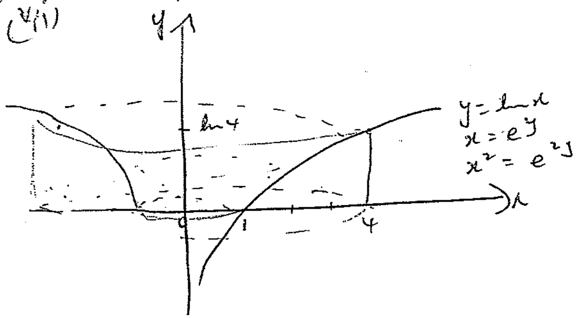
$= \frac{1}{2}\left[1 - \ln 2 - (1 - \ln 2)\right]$

$= \frac{1}{2}(1 - \ln 2 - 1 + \ln 2)$

$= \frac{1}{2}(0)$

$= 0$

(v)(ii)



$\frac{1}{2}$

$$V = \pi r^2 h - \pi \int_0^{\ln 4} e^{2y}\, dy$$

$$= \pi . 4^2 . \ln 4 - \pi \left[\frac{e^{2y}}{2} \right]_0^{\ln 4}$$

$$= 16\pi \ln 4 - \frac{\pi}{2}\left[e^{2\ln 4} - e^0 \right]$$

$$= 16\pi \ln 4 - \frac{\pi}{2}\left[e^{\ln 4^2} - 1 \right]$$

$$= 16\pi \ln 4 - \frac{\pi}{2}(16 - 1)$$

$$= 16\pi \ln 4 - \frac{\pi}{2}(15)$$

$$= 16\pi \ln 4 - \frac{15\pi}{2}.$$

$\frac{}{12}$

Question 4

(i) (a) In Δs AXM, YBM

(i) ∠AMX = ∠YMB (vertically opposite angles)

(ii) ∠XAM = ∠BYM (angles subtended by arc XB are equal)

(iii) ∠AXM = ∠YBM (angles subtended by arc AY are equal)

∴ ΔAXM ||| ΔYBM (equiangular)

(b)

$$\frac{XM}{BM} = \frac{AM}{YM} \quad \text{(corresponding sides of similar Δ's are in proportion)}$$

$$\frac{8}{\frac{1}{2}r} = \frac{r + \frac{1}{2}r}{6}$$

$$48 = \frac{1}{2}r\left(1\tfrac{1}{2}r\right)$$

$$48 = \frac{1}{2}r \times \frac{3}{2}r$$

$$48 = \frac{3r^2}{4}$$

$$16 = \frac{r^2}{4}$$

$$16 \times 4 = r^2$$

$$4 \times 2 = r$$

$r = 8$ cm.

11) (a) $\angle APC = \angle ABC$ (angles at the circumference standing on arc AC are equal)

(b) let $\angle APC = \angle ABC = \alpha$

$\angle ADE = \alpha$ (exterior angle of cyclic quad ADEB equals interior remote angle)

$\angle APK = 180^\circ - \alpha$ ($\angle CPK$ is a straight angle)

$\therefore \angle APK + \angle ADE = 180^\circ - \alpha + \alpha$

$= 180^\circ$.

$\therefore$ ADKP is a cyclic quad (opposite angles are supplementary)