Randwick Girls High School
Mathematics Extension 2
Assessment Task 2
2006

**Time allowed 1 hour** Name________________
**Calculators may be used**
**Marks may be deducted for badly Arranged work**
**Examiner V. Jaggar.**

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|---|---|
| **Question 1** | |
| **Question 2** | |
| **Question 3** | |
| **Total** | |

**Question 1 (14 marks)**

a) Resolve $\frac{x^2+1}{x^3-1}$ into partial fractions. 5

b) i) Express $\sqrt{3}-i$ in modulus/argument from. 2
ii) Hence evaluate $(\sqrt{3}-i)^6$

c) Find the coordinates of the point of contact between the hyperbola 4

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ and the tangent to the hyperbola } Ax+By+C=0.$$

d) Differentiate implicitly $(x^2+y^2)^2=2(x^2-y^2)$ to find an expression for $\frac{dy}{dx}$. 3

**Question 2 (15 marks)**

a) Find the roots of $P(x)=x^3+3x^2-4$, given that P(x) has a double root. 4

b) Express $P(x)= x^4-x^3-5x^2-x-6$ as a product of irreducible factors over Real Numbers 2

**Question 2 (15 marks) continued**

c) If $\alpha, \beta, \gamma$ are the roots of $x^3 + px + q = 0$, find the value of; 5

i) $(\alpha - 1), (\beta - 1), (\gamma - 1)$

ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

iii) $\alpha^3 + \beta^3 + \gamma^3$

iv) Find the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, $\alpha^3, \beta^3, \gamma^3$. 4

**Question 3 (16 marks)**

a) i) Show that $4x^2 + 9y^2 + 16x + 18y - 11 = 0$ represents an ellipse. 2

ii) Find the eccentricity and hence the coordinates of its foci and the equations of its directrices. Sketch the ellipse and label fully. 4

c) The tangent at $P\left(kp, \frac{k}{p}\right)$ to the hyperbola $xy = k^2$ meets the x-axis at Q. The normal to the hyperbola at P meets the curve again at M.

i) Find the coordinates of Q in terms of $p$. 4

ii) Find the area of the triangle PQM in terms of. P and Prove that the area is a minimum when p = ± 1 . 6

The end

R.G.H.S — SOLUTIONS EXT 2. — Apr. 2006

## Quest (1)

(a) $\dfrac{x^2+1}{(x-1)(x^2+x+1)} \equiv \dfrac{A}{x-1} + \dfrac{Bx+c}{x^2+x+1}$

$\therefore x^2+1 \equiv Ax^2+Ax+A+Bx^2+(c-B)x-c$

$x^2+1 \equiv (A+B)x^2+(A+c-B)x+A-C.$

1) $A+B=1 \longrightarrow B=1-A$
2) $A+C-B=0$
3) $A-C=1 \longrightarrow C=A-1$

Sub: into (2) $\rightarrow A+(A-1)-(1-A)=0$

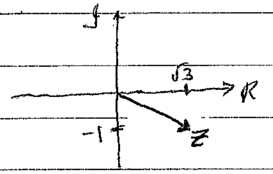
$\therefore 3A=2 \quad A=\frac{2}{3}$

$\therefore B=\frac{1}{3}$

$C=-\frac{1}{3}$

$\therefore \dfrac{x^2+1}{x^3-1} \equiv \dfrac{2}{3(x-1)} + \dfrac{x+1}{3(x^2+x+1)}$

(b)




i) $|z|=\sqrt{(\sqrt{3})^2+1^2}=2.$ Let $\theta=\operatorname{Arg} z$

$\tan^{-1}\theta=\frac{1}{\sqrt{3}} \rightarrow \operatorname{Arg} z=\theta=-\frac{\pi}{6}.$

$\therefore z=2(\cos\frac{-\pi}{3}+i\sin\frac{-\pi}{3})$

ii) $\therefore z^6=64[\cos(-2\pi)+i\sin(-2\pi)]=64.$

(c) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ has Tangent $\boxed{\dfrac{x.x_0}{a^2}-\dfrac{y.y_0}{b^2}=1}$ at $T(x_0,y_0)$

Re-arrange $Ax+By+C=0$ to this form ↑ gives — COMPARE!

$\dfrac{Ax}{c}+\dfrac{By}{c}+1=0 \longrightarrow \boxed{\dfrac{-Ax}{c}-\dfrac{By}{c}=1}$

$\Rightarrow \dfrac{x_0}{a^2}=-\dfrac{A}{c}$ and $\dfrac{y_0}{b^2}=\dfrac{By}{c}.$

$\therefore x_0=-\dfrac{Aa^2}{c}. \qquad \therefore y_0=\dfrac{Bb^2}{c}.$

(d) $2(x^2+y^2)^1\times(2x+2y.y')=2(2x-2y.y')$

$\therefore 4[(x^2+y^2)(x+2y.y')]=4(x-y.y')$

$\therefore x^3+2x^2y.y'+xy^2+2y^3y'-x+y.y'=0$

$\therefore (2x^2y+2y^3+y)y'=x-x^3-xy^2$

$y'=\dfrac{x(1-x^2-y^2)}{y(2x^2+2y^2+1)}$

## Quest (2)

(a) $P(x)=x^3+3x^2-4$

$P'(x)=3x^2+6x$
$=3x(x+2)=0$ when $x=0$, $x=-2$

$\therefore P(x)=(x+2)^2(ax+b)$

$\therefore P(x)=(x+2)^2(x-1)$

$\therefore$ Roots are $x=-2,-2,1$

$$\begin{array}{r|l} & x-1 \\ x^2+4x+4 & x^3+3x^2\quad -4 \\ & \underline{x^3+4x^2+4x} \\ & -x^2-4x-4 \end{array}$$

OR use $\alpha+\beta+\gamma=3 \quad \therefore -2-2+\gamma=3$

(b) $P(3)=81-27-45-3-6=0$
& $P(-2)=16+8-20+2-6=0$
$\therefore (x-3)(x+2)$ is a factor of $P(x)$

$$\begin{array}{r|l} & x^2\quad +1 \\ x^2-x-6 & x^4-x^3-5x^2-x-6 \\ & \underline{x^4-x^3-6x^2} \\ & x^2-x-6 \end{array}$$

$\therefore P(x)=(x-3)(x+2)(x^2+1)$

(c) i) $(\alpha-1)(\beta-1)(\gamma-1)=\alpha\beta\gamma-(\alpha\beta+\alpha\gamma+\beta\gamma)+(\alpha+\beta+\gamma)-1$

$=(-\frac{d}{a})-(\frac{c}{a})+(-\frac{b}{a})-1$

$=-q-p+0-1$

$=-(p+q+1)$

$a=1$, $b=0$, $c=p$, $d=q$

ii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}=\dfrac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}=\dfrac{(c/a)}{(-d/a)}=\dfrac{p}{-q}$

iii) $\alpha,\beta,\gamma$ roots $\Rightarrow$
$\alpha^3+p\alpha+q=0$
$\beta^3+p\beta+q=0$
$\gamma^3+p\gamma+q=0$
ADD $\therefore$

$\therefore \alpha^3+\beta^3+\gamma^3+p(\alpha+\beta+\gamma)+3q=0$

$\therefore \alpha^3+\beta^3+\gamma^3=-3q-p(0)=-3q.$

(iv)

Qu(2) cont'.

(iv) Let $P_6(x)=0$ with roots $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma},\alpha^3,\beta^3,\gamma^3$ be $P_6(x)=Q_3(x)\times R_3(x)$

— where $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ are roots of $Q_3(x)=0$ & $\alpha^3,\beta^3,\gamma^3$ are roots of $R_3(x)=0$

For $Q(x)=0$ let $x=\frac{1}{X}$ $\rightarrow (\frac{1}{X})^3+p(\frac{1}{X})+q=0$ $\quad(\times X^3)$

$\rightarrow 1+pX^2+qX^3=0.$

For $R(x)=0$ let $x=\sqrt[3]{X}$

$\rightarrow (\sqrt[3]{X})^3+p(\sqrt[3]{X})+q=0$

$\rightarrow X+q=-p\sqrt[3]{X}$ ← cube both sides...

$\therefore X^3+3qX^2+3q^2X+q^3=-p^3X$ or $X^3+3qX^2+(3q^2+p^3)X+q^3=0$

$\therefore P_6(X)=(X^3+3qX^2+(3q^2+p^3)X+q^3)(qX^3+pX^2+1)=0.$

$=qX^6+3q^2X^5+3q^3X^4+p^3qX^4+q^4X^3$

$+pX^5+3pqX^4+3pq^2X^3+p^4X^3+pq^3X^2$

$+X^3+3qX^2+(3q^2+p^3)X+q^3$

$\Rightarrow qX^6+(3q^2+p)X^5+(3q^3+p^3q+3pq)X^4+(q^4+3pq^2+p^4+1)X^3$

$+(pq^3+3q)X^2+(3q^2+p^3)X+q^3=0$

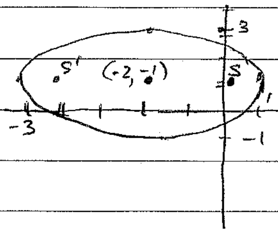
Quest(3).

(a) i) $4x^2+16x+9y^2+18y=11$

$4(x^2+4x\quad)+9(y^2+2y\quad)=11$

$4(x+2)^2+9(y+1)^2=11+16+9$

$\Rightarrow \frac{(x+2)^2}{9}+\frac{(y+1)^2}{4}=1$



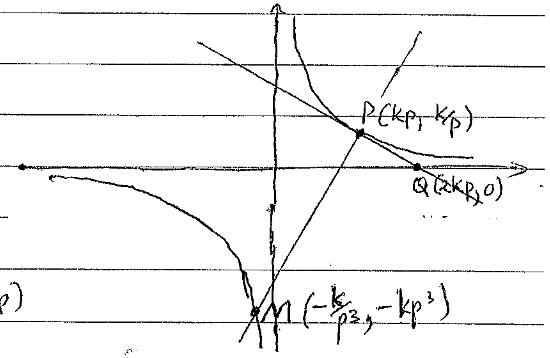

ii) $b^2=a^2(1-e^2)$ $[a=3 \quad b=2]$ $\quad S=(ae-2,-1)=(\sqrt{5}-2,-1)$ $\quad S'=(-ae-2,-1)=(-\sqrt{5}-2,-1)$

$\therefore 4=9(1-e^2)$

$\therefore e^2=1-\frac{4}{9}=5/9$ $\quad$ Directrices: $x=\frac{a}{e}-2$ & $-\frac{a}{e}-2$

$e=\sqrt{5}/3$ $\quad$ ie $x=\frac{9}{\sqrt{5}}-2$ & $x=-\frac{9}{\sqrt{5}}-2.$

Qu(3) con't




$xy=k^2\rightarrow y=\frac{k^2}{x}$

$\rightarrow y'=-\frac{k^2}{x^2}$

at P; $y'=\frac{-k^2}{k^2p^2}=-\frac{1}{p^2}$

$\therefore$ Equ. Tangent is: $y-\frac{k}{p}=-\frac{1}{p^2}(x-kp)$

$\therefore p^2y-kp=kp-x.$

or $\boxed{x=2kp-p^2y}$ for Q subst. $y=0 \Rightarrow x=2kp$ : $Q=(2kp,0)$

Equ^n of Normal is: $y-\frac{k}{p}=p^2(x-kp)$

$y=p^2x-kp^3+\frac{k}{p}$ → Solve with $xy=k^2$

$\Rightarrow p^2x^2-kp^3x+\frac{kx}{p}=k^2$

$\Rightarrow p^2x^2+(\frac{k}{p}-kp^3)x-k^2=0$

$\Delta=b^2-4ac=(\frac{k}{p}-kp^3)^2+4p^2k^2$

$=\frac{k^2}{p^2}+k^2p^6-2p^2k^2+4p^2k^2$

$=\frac{k^2}{p^2}+2p^2k^2+k^2p^6$

$=(\frac{k}{p}+kp^3)^2$

$\therefore \cancel{x}$

$\therefore x=\frac{kp^3-\frac{k}{p}\pm(\frac{k}{p}+kp^3)}{2p^2}=\frac{2kp^3}{2p^2}$ or $\frac{-2k}{2p^3}$

ie $x=kp$ or $x=-\frac{k}{p^3}$ $\quad y=-kp^3$

$\therefore MP=(p+\frac{1}{p^3})\sqrt{p^4+1}$

$PQ=k/p\sqrt{p^4+1}$

$\therefore$ Area $=\frac{1}{2}\frac{k}{p}(p+\frac{1}{p^3})(p^4+1)=\frac{k}{2}(1+\frac{1}{p^4})(p^4+1)=\frac{k}{2p^4}(p^4+1)^2$ u²