RANDWICK GIRLS' HIGH SCHOOL

NAME: ______________

TEACHER: ______________

YEAR 12 ASSESSMENT TASK

Term 2, 2008

Mathematics Extension 1

Time Allowed: 1 hour **Examiner: D. Posener**

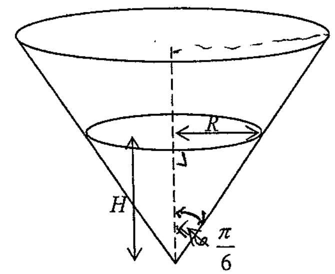
GENERAL INSTRUCTIONS:

- All questions may be attempted
- Standard Integral Tables are supplied
- Department of Education approved calculators and templates are permitted
- In every question, show all necessary working
- Marks may not be awarded for careless or badly arranged work

QUESTION	MARKS
1	/11
2	/10
3	/8
4	/8
5	/10
6	/9
Total	/56

Question 1

(a) The semi-vertical angle of a conical funnel is $\frac{\pi}{6}$. Water is flowing out of the funnel at a rate of $5\text{cm}^3\, s^{-1}$.



(i) Show that if R is the radius of the water surface and H is the depth of water at any instant, then $H = \sqrt{3}\, R$ **2**

(ii) Find the rate of decrease of the surface radius when the depth of water in the funnel is 10cm. **3**

(b) Newton's Law of Cooling states that the rate of cooling of an object is proportional to the difference between the temperature T of the object and the temperature S of the surrounding medium. That is, $\frac{dT}{dt} = k(S - T)$ where t is the time elapsed and k is the "rate constant".

(i) Verify that $T = S + Ae^{-kt}$, where A is constant, is a solution to the above differential equation. **2**

(ii) A body whose temperature is 150°C is immersed in a liquid kept at a constant temperature of 70°C. In 40 minutes the temperature of the body falls to 90°C.

a. Determine the value of k. **2**

b. How long altogether will it take for the temperature of the body to fall to 76°C? Give your answer correct to the nearest minute. **2**

Question 2

A particle moves so that its velocity vms^{-1} at position x is given by $v^2 = 5 + 4x - x^2$.

(a) Show that the motion is simple harmonic. **2**

(b) State **2**

(i) The centre
(ii) The period

(c) Find the maximum speed. **2**

(d) Find the amplitude. **2**

(e) Find the maximum acceleration. **2**

Question 3

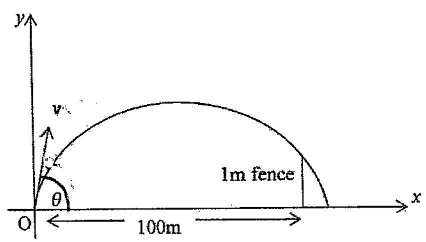
(a) Simplify $\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{3}\right)$ to find exact value. **4**

(b) (i) Show $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ **3**

(ii) Sketch $y = \sin^{-1} x + \cos^{-1} x$ **1**

Question 4

A "six" is scored in a cricket game when the ball is hit over the boundary fence on the full as in the diagram. A ball is hit from O with velocity $v = 32ms^{-1}$ at an angle θ to the horizontal and towards the 1 metre high boundary fence, 100 metres away.



(i) Derive the equations of motion for the ball in flight using axes as in the diagram. (Air resistance is to be neglected and the acceleration due to gravity is taken as $10ms^{-2}$) **2**

(ii) Show that the ball just clears the boundary fence when $50000\tan^2\theta - 102400\tan\theta + 51024 = 0$ **3**

(iii) In what range must θ lie for a "six" to be scored? **3**

Question 5

The normal at any point $P\ (2ap, ap^2)$ on the parabola $x^2 = 4ay$ cuts the y-axis at Q and is produced to a point R such that $PQ = QR$.

(a) Show that R has coordinates $(-2ap, ap^2 + 4a)$ **5**

(b) Show that the locus of R is another parabola. **3**

(c) State the coordinates of the vertex and focus of this second parabola. **2**

Question 6

(a) Find the term independent of x in the expansion of $\left(\frac{x^2}{2} - \frac{3}{x}\right)^9$ **4**

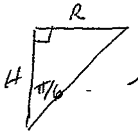
(b) By considering the expansion of $(1 + x)^n$ and giving x an appropriate numerical value, prove $2^nC_2 + 6^nC_3 + 12^nC_4 + \cdots n(n-1)^nCn = n(n-1)2^{n-2}$ **5**

Solutions

2P Q 1, 2, 6

TM Q 3, 4, 5.

Q1 (a) (i)



$\tan\frac{\pi}{6} = \frac{R}{H}$ ✓ (1)

$\frac{1}{\sqrt{3}} = \frac{R}{H}$

$H = \sqrt{3}R$ ✓ (1)

(ii)

$V = \frac{1}{3}\pi R^2 H$

$= \frac{1}{3}\pi R^2 . R\sqrt{3}$

$= \frac{1}{3}\pi R^3\sqrt{3}$

$= \frac{1}{3}\sqrt{3}\,\pi R^3$

$\frac{dV}{dR} = 3.\frac{1}{3}\sqrt{3}\,\pi R^2$

$= \sqrt{3}\,\pi R^2$ ✓

$\frac{dV}{dR} = \frac{dV}{dt} \times \frac{dt}{dR}$

$\sqrt{3}\pi R^2 = -5 \times \frac{dt}{dR}$ ✓

$H = 10$:
$10 = \sqrt{3}R$
$R = \frac{10}{\sqrt{3}}$

$\sqrt{3}\pi . \frac{\cancel{100}^{20}}{3} \times \frac{1}{-\cancel{5}} = \frac{dt}{dR}$

$\frac{dt}{dR} = -\frac{\sqrt{3}}{3}.20\pi$

$= -\frac{20\sqrt{3}\pi}{3}$

$\therefore \frac{dR}{dt} = -\frac{3}{20\sqrt{3}\pi} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{-\cancel{3}\sqrt{3}}{20.\cancel{3}.\pi}$

$= -\frac{\sqrt{3}}{20\pi}$ cm s^{-1}. ✓

$-\frac{\sqrt{3}}{20\pi} \times \frac{\sqrt{3}}{\sqrt{3}} = -\frac{3}{20\pi\sqrt{3}}$

(b). (i). Consider $T = S + Ae^{-kt}$

$\therefore \frac{dT}{dt} = 0 + Ae^{-kt}.-k$

$= -kAe^{-kt}$

$= -k(T - S)$

$= k(S - T)$ ✓

(ii) (α) $t = 0$, $T = 150$

$T = 70 + Ae^{-kt}$

$150 = 70 + Ae^{\cancel{0}}$

$80 = A$

$\therefore T = 70 + 80e^{-kt}$

$t = 40$, $T = 90$

$90 = 70 + 80e^{-40k}$

$20 = 80e^{-40k}$

$\frac{20}{80} = e^{-40k}$

$\frac{1}{4} = e^{-40k}$

$\ln\frac{1}{4} = \ln e^{-40k}$

$\ln\frac{1}{4} = -40k \ln e$

$\frac{\ln\frac{1}{4}}{-40} = k$ ✓

$k = 0.03465$

$\doteqdot 0.03$

$T = 70 + 80e^{-kt}$

(β) $t = ?$, $T = 76°$

$76 = 70 + 80e^{-0.03t}$

$6 = 80e^{-0.03t}$

$\frac{6}{80} = e^{-0.03t}$

$\ln\left(\frac{6}{80}\right) = \ln e^{-0.03t}$

$\ln\frac{6}{80} = -0.03t \ln e$

$\frac{\ln\left(\frac{6}{80}\right)}{-0.03} = t$ ✓

$t = 74.739$ min

$= 75$ min.

$\frac{\cancel{6}\,3}{\cancel{80}\,40}$

Question 2:

(a) $v^2 = 5 + 4x - x^2$

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{5}{2} + 2x - \frac{1}{2}x^2\right)$

$= 0 + 2 - x$

$= 2 - x$ ✓

$= -1^2(x-2)$

2 $= -n^2(x-h)$

$\therefore$ SHM with $n=1$ and $h=2$.

1 (b) (i) Centre is at $x=2$

1 (ii) Period $= \frac{2\pi}{n} = \frac{2\pi}{1} = 2\pi$ sec.

(c) Max. speed occurs at equilibrium position.

$x = 2$.

2 $v^2 = 5 + 4(2) - (2)^2 = 9$

$v = \pm 3$.

$\therefore$ max. speed $= 3\,ms^{-1}$.

(d) amplitude at $v=0$

$v^2 = 0$

$5 + 4x - x^2 = 0$

2 $x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$\therefore x = 5$ or $x = -1$

$\therefore$ Amplitude $= 5 - 2 = 3m$.

2 (e) acc. is max. at end points.

If $x = 5$, $a = 2 - 5 = -3$

/10 $\therefore$ max. acceleration is $3\,ms^{-2}$.

Question 3:

(a) $\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{3}\right)$

let $\tan^{-1}\left(\frac{1}{2}\right) = \alpha$ and $\tan^{-1}\left(\frac{1}{3}\right) = \beta$.

$\therefore \tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$

LHS $= \alpha - \beta$

$\tan$ LHS $= \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$= \frac{\frac{1}{2} - \frac{1}{3}}{1 + \frac{1}{2}\cdot\frac{1}{3}}$

$= \frac{1/6}{7/6}$

$= \frac{1}{7}$

$\therefore$ LHS $= \tan^{-1}\left(\frac{1}{7}\right)$

$\therefore \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\frac{1}{7}$.

(b) (i) Let $\alpha = \sin^{-1}x$ and $\beta = \cos^{-1}x$

$\therefore x = \sin\alpha$ and $x = \cos\beta$

$\therefore \sin\alpha = \cos\beta$

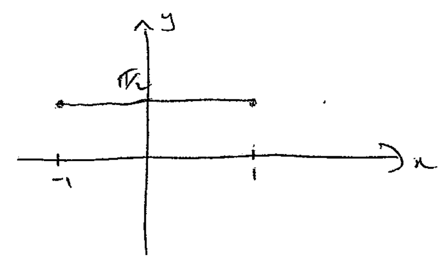
$\sin\alpha = \sin\left(\frac{\pi}{2} - \beta\right)$

$\alpha = \frac{\pi}{2} - \beta$

$\therefore \alpha + \beta = \frac{\pi}{2}$

$\therefore \sin^{-1}x + \cos^{-1}x = \pi/2$

(ii)



Q4. (i) $\frac{d^2x}{dt^2} = 0$ $\frac{d^2y}{dt^2} = -10$

when $t=0$, $x=0$, $y=0$, $\frac{dx}{dt} = V\cos\theta = 32\cos\theta$ $\frac{dy}{dt} = V\sin\theta = 32\sin\theta$.

$\frac{dx}{dt} = c_1$ $\frac{dy}{dt} = -10t + c_2$

$32\cos\theta = c_1$ $32\sin\theta = c_2$

$\therefore \frac{dx}{dt} = 32\cos\theta$ $\therefore \frac{dy}{dt} = -10t + 32\sin\theta$

$x = 32\cos\theta t + c_3$ $y = \frac{-10t^2}{2} + 32\sin\theta t + c_4$

$c_3 = 0$ $c_4 = 0$

$x = 32\cos\theta t$ $y = -5t^2 + 32\sin\theta t$.

(ii) from $x = 32\cos\theta t$

$t = \frac{x}{32\cos\theta}$

into $y = -5t^2 + 32\sin\theta t$

$= \frac{-5 \cdot x^2}{32^2\cos^2\theta} + 32\sin\theta \cdot \frac{x}{32\cos\theta}$

$= -\frac{5x^2}{1024}\sec^2\theta + x\tan\theta$

$= -\frac{5x^2}{1024}(1+\tan^2\theta) + x\tan\theta$

$(100, 1)$ satisfies this equation

$1 = \frac{-5 \cdot 10000}{1024}(1+\tan^2\theta) + 100\tan\theta$

$1024 = -50000(1+\tan^2\theta) + 102400\tan\theta$

$\therefore 1024 = -50000 - 50000\tan^2\theta + 102400\tan\theta$

$\therefore 50000\tan^2\theta - 102400\tan\theta + 51024 = 0$.

(iii) $\tan\theta = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{102400 \pm \sqrt{102400^2 - 4 \times 50000 \times 51024}}{2 \cdot 50000}$

$= \frac{102400 \pm \sqrt{280960000}}{100000}$

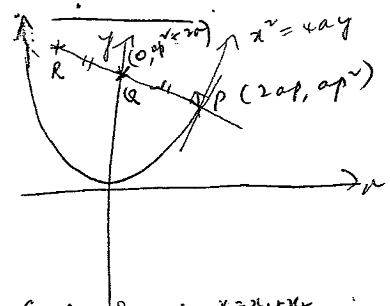
$\tan\theta = \frac{102400 + 16761.86147}{100000}$ or $\tan\theta = \frac{102400 - 16761.86147}{100000}$

$\theta = 40°35'$ or $50°$.

Range is [illegible]

Question 5.

(a)



x coords of Q : $x = \frac{x_1 + x_2}{2}$

$0 = \frac{x_1 + 2ap}{2}$

$0 = x_1 + 2ap$

$x_1 = -2ap$

$\therefore$ R has x-coord $-2ap$.

$R(-2ap, ap^2 + 4a)$

Consider $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

$\therefore$ slope of tangent at $P = \frac{2ap}{2a} = p$

$\therefore$ slope of normal at $P = -\frac{1}{p}$

eqn of normal is:

$y - y_1 = m(x - x_1)$

$y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$.

normal cuts y-axis when $x = 0$

$0 + py = ap^3 + 2ap$

$y = ap^2 + 2a$.

y coord of Q is

$y = \frac{y_1 + y_2}{2}$

$ap^2 + 2a = \frac{y_1 + ap^2}{2}$

$2ap^2 + 4a = y_1 + ap^2$

$y_1 = 2ap^2 - ap^2 + 4a$

$= ap^2 + 4a$.

(b) $x = -2ap$

$\frac{x}{-2a} = p$

Sub into $y = ap^2 + 4a$

$= a\left(\frac{x^2}{4a^2}\right) + 4a$

$y = \frac{x^2}{4a} + 4a$

$4ay = x^2 + 16a^2$

$x^2 = 4ay - 16a^2$

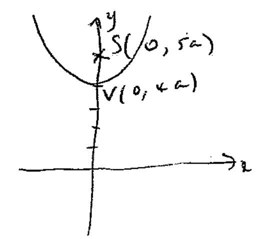
$x^2 = 4a(y - 4a)$

which is a parabola

(c) $(x-0)^2 = 4a(y-4a)$

$(x-h)^2 = 4a(y-k)$

$\therefore V(0, 4a)$ $S(0, 5a)$.



Question 6:

(a) $T_{k+1} = {}^9C_k \left(\frac{x^2}{2}\right)^{9-k} \left(-\frac{3}{x}\right)^k$

$= A x^{18-2k} \cdot x^{-k}$ ✓

$= A x^{18-3k}$

${}^9C_6 \frac{3^6}{2^3}$

$\therefore 18 - 3k = 0$

$3k = 18$

$k = 6.$ ✓

$\therefore$ Term indept of x is $T_7 = {}^9C_6 \left(\frac{x^2}{2}\right)^3 \left(-\frac{3}{x}\right)^6$ ✓

$= 84 \cdot 2^{-3} \cdot (-3)^6 \cdot x^6 \cdot x^{-6}$

$= 84 \cdot \frac{1}{8} \cdot +729$

$= +7654 \cdot 5.$ ✓

4

(b) $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots + {}^nC_n x^n$ ✓

Diff. b.s. wrt x

$n(1+x)^{n-1} = 1 \cdot {}^nC_1 x^0 + 2 \cdot {}^nC_2 x^1 + 3 \cdot {}^nC_3 x^2 + 4\, {}^nC_4 x^3 + \dots + n \cdot {}^nC_n x^{n-1}$ ✓

$n(1+x)^{n-1} = 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 x^1 + 3 \cdot {}^nC_3 x^2 + 4\, {}^nC_4 x^3 + \dots + n \cdot {}^nC_n x^{n-1}$

Diff b.s wrt x again ✓

$n(n-1)(1+x)^{n-2} = 2 \cdot {}^nC_2 x^0 + 6 \cdot {}^nC_3 x^1 + 12\, {}^nC_4 x^2 + \dots$

$\dots + n(n-1) \cdot {}^nC_n x^{n-2}$

5

let $x = 1$ ✓

$n(n-1)(2)^{n-2} = 2 \cdot {}^nC_2 + 6 \cdot {}^nC_3 + 12\, {}^nC_4 + \dots + n(n-1) \cdot {}^nC_n$ ✓