Randwick Girls High School

Name: ....

Class/Teacher: ....

2014 HSC Assessment Task 2

Year 12 Mathematics Extension 2

Time allowed : 2 hours

- Use blue or black pen.
- Approved calculators may be used.
- Start each question on a new page.
- All necessary working should be shown.

Question	Marks
Multiple choice	/5
Q. 1 Graphs	/25
Q. 2 Conics	/19
Total	/49

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq 1;\quad x\neq 0,\quad \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x>0$

Multiple Choice Questions (5 marks) Use a SEPARATE sheet of paper

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1) The range of the function $y = \frac{1}{2+x^2}$ is:

a) $y \geq 0$

b) $y > 0$

c) $-\frac{1}{2} \leq y < 0$

d) $0 < y \leq \frac{1}{2}$

2) Which conic section has the eccentricity e = 1?

a) parabola

b) hyperbola

c) ellipse

d) circle

3) Given the equation of an ellipse is $\frac{x^2}{64} + \frac{y^2}{9} = 1$, what are the parametric equations?

a) $x = 3cos\theta, y = 8sin\theta$

b) $x = 8cos\theta, y = 3sin\theta$

c) $x = 3sin\theta, y = 8cos\theta$

d) $x = 8sin\theta, y = 3cos\theta$

4) The graph of $y = (x - a)^2 + b$ is reflected about the y-axis. Which of the following is a correct statement about the reflected image?

a) The image has a minimum turning point at $(a, -b)$

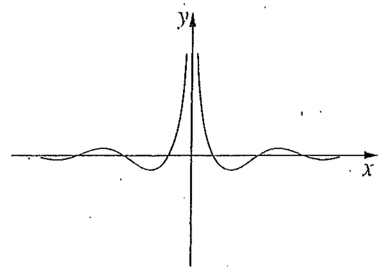
b) The image has a maximum turning point at $(-a, b)$

c) The image has a minimum turning point at $(-a, b)$

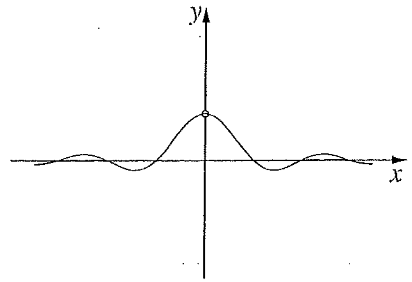
d) The image has a maximum turning point at $(a, -b)$

5) Which diagram best represents the graph $y = \frac{sinx}{x}$?

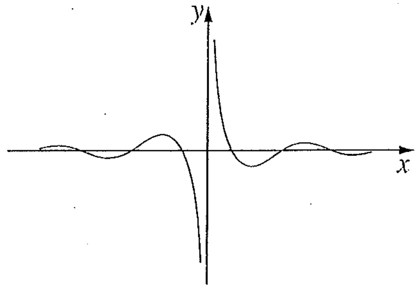
a)



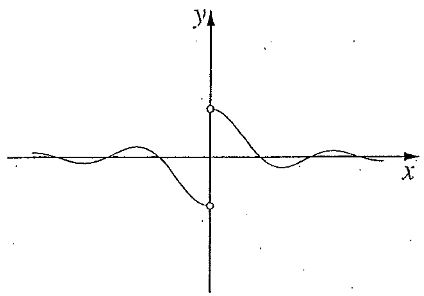
b)



c)



d)



Question 1 (25 marks) Use a SEPARATE sheet of paper **Marks**

a) i) Sketch the following graphs for $-2\pi \leq x \leq 2\pi$ on the same number plane without using calculus: **2**

$$y = \frac{x}{2} \text{ and } y = \sin(x).$$

ii) On a separate number plane sketch, for $-2\pi \leq x \leq 2\pi$, the graph of: **2**

$$y = \frac{2\sin(x)}{x}$$

b) Given $y = \frac{x^3}{x^2-4}$

i) Find the coordinates of all stationary points. **2**

ii) Find x-intercepts, y-intercepts and all asymptotes. **2**

iii) Hence, sketch the curve $y = \frac{x^3}{x^2-4}$ **2**

c) Sketch the following curves:

i) $y = \ln(x+1)$ **2**

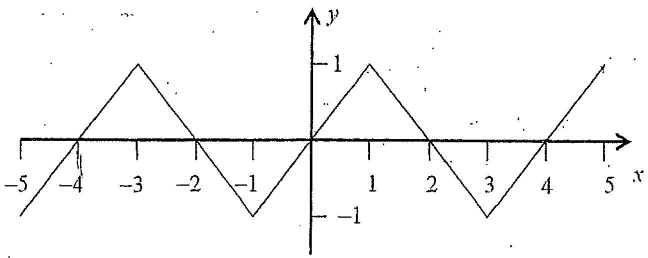
ii) $y = \ln|x+1|$ **1**

iii) $y = |\ln(x+1)|$ **1**

iv) $y = \frac{1}{\ln(x+1)}$ **3**

Marks

d) The diagram below is a sketch of the function $y = h(x)$ for $-5 \leq x \leq 5$.
On separate diagrams, sketch each of the following:



i) $y = h(x+1)$ **1**

ii) $y = \frac{1}{h(x)}$ **2**

iii) $y = h(|x|)$ **1**

iv) $y = \sqrt{h(x)}$ **2**

v) $y = h(\sqrt{x})$ **2**

Question 2 (19 marks) Use a SEPARATE sheet of paper **Marks**

a) i) For the hyperbola, $\frac{x^2}{9} - \frac{y^2}{16} = 1$, find its eccentricity e. **2**

ii) Hence, neatly sketch the above hyperbola, clearly showing the vertices, foci, directrices and asymptotes. **4**

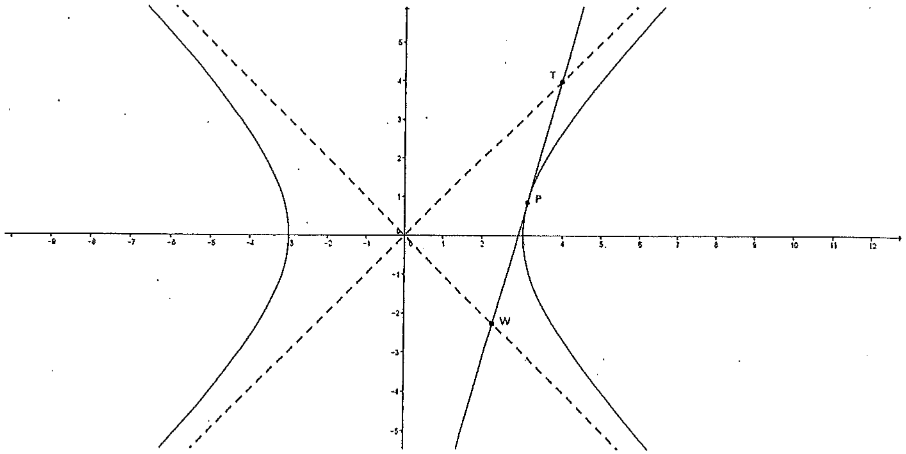
b) An ellipse has the equation $\frac{x^2}{100} + \frac{y^2}{75} = 1$.

i) Sketch the above ellipse, clearly showing on your diagram the coordinates of the foci and the equation of each directrix. **4**

ii) Show that the equation of the normal to the ellipse is: **2**

$$4x - 2y = 5 \text{ at point } P\left(5, 7\tfrac{1}{2}\right)$$

c) The point $S(3e, 0)$ is a focus on the hyperbola $x^2 - y^2 = 9$. The tangent to the hyperbola, at the point $P(p, q)$, meets the asymptotes of the hyperbola at T and W.



i) Show that the equation of the tangent TW is given by $px - qy = 9$. **2**

ii) Show that the gradient of the line through SW is given by **2**

$$m_{SW} = \frac{3}{e(p+q) - 3}$$

iii) By letting $\angle WST = \theta$, find the value of $\tan\theta$. **3**

Little Chow

(5)

Multi

1 D

2 A

3 B

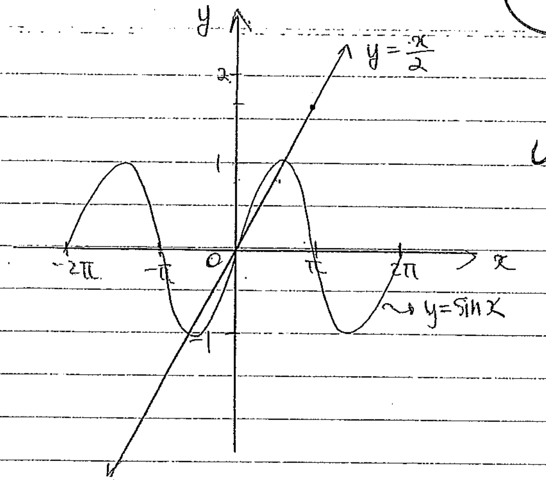
4 C

5 B

Little Chow

(23)

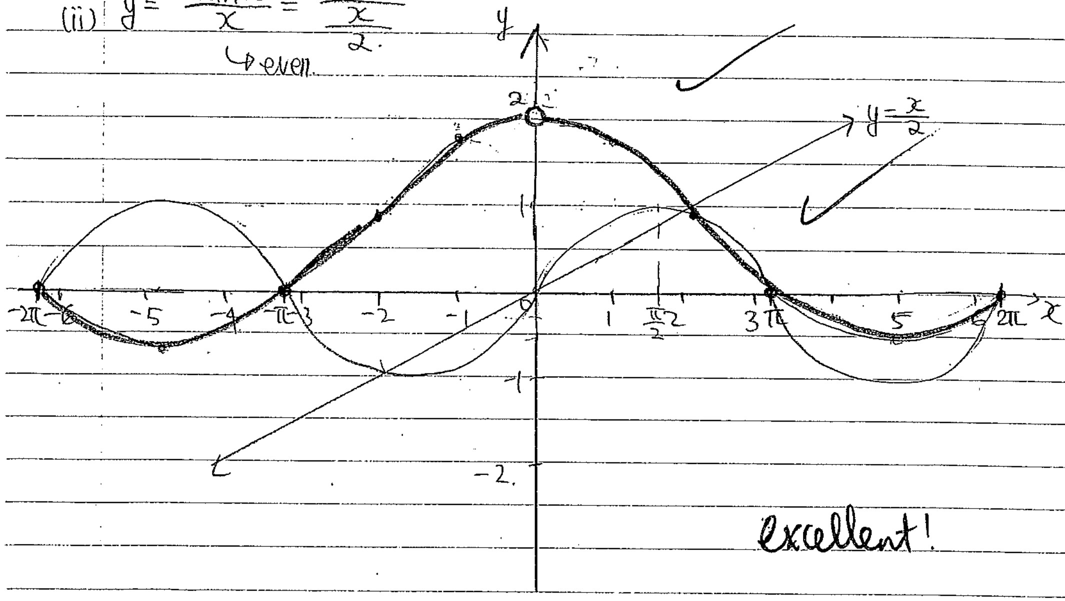
(a)



$\frac{2\sin(-x)}{-x} =$

(ii) $y = \frac{2\sin x}{x} = \frac{\sin x}{\frac{x}{2}}$

↳ even



excellent!

$\lim_{x\to 0} \frac{2\sin x}{x} = 2.$

$(x \neq \pm 2)$

(b) $y = \frac{x^3}{x^2-4}$ → odd

(i) $\frac{dy}{dx} = \frac{(x^2-4)\cdot 3x^2 - x^3(2x)}{(x^2-4)^2}$

$= \frac{3x^4 - 12x^2 - 2x^4}{(x^2-4)^2}$

$= \frac{x^4-12x^2}{(x^2-4)^2}$

stat pt : $\frac{dy}{dx} = 0$

$\frac{x^4-12x^2}{(x^2-4)^2} = 0 \quad (x \neq \pm 2)$

$x^2(x^2-12)=0$

$x=0$ or $x^2=12$

$x = \pm 2\sqrt{3}$

① $x=0$, $y=0$
hz ZNF $(0,0)$

+ - - 0 - - + $\frac{dy}{dx}$
$-2\sqrt{3}$ 0 $2\sqrt{3}$
max ZNF min

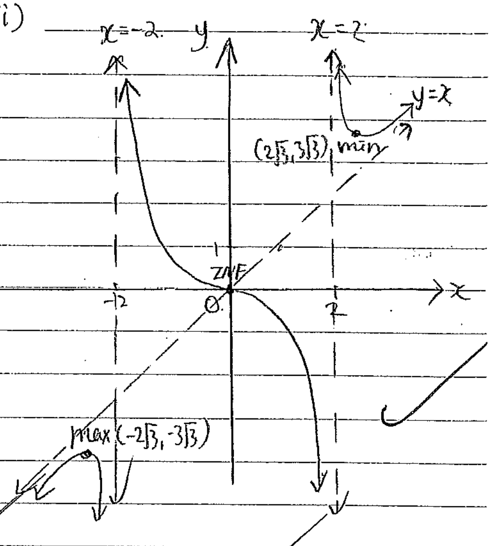
② $x = 2\sqrt{3}$,
$y = \frac{(2\sqrt{3})^3}{12-4} = \frac{24\sqrt{3}}{8} = 3\sqrt{3}$

∴ min$(2\sqrt{3}, 3\sqrt{3})$

③ $x=-2\sqrt{3}$, $y=-3\sqrt{3}$
max$(-2\sqrt{3}, -3\sqrt{3})$

(ii) $y=0$, $x^3=0$, $x=0$
$(0,0)$ → x & y-int.
vertical asymp: $x = \pm 2$
oplique " : $y=x$

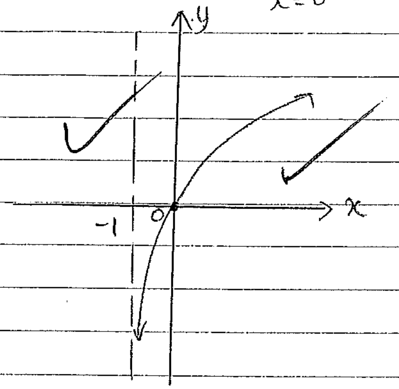
(iii)



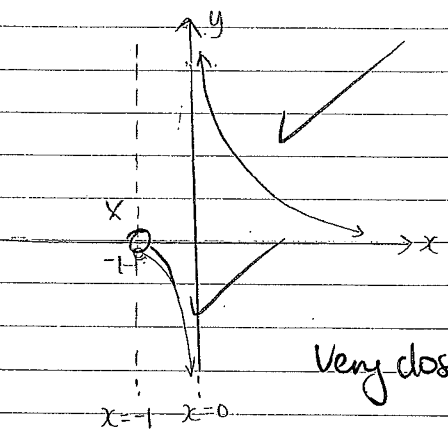
$y = \frac{x^3}{x^2-4}$
$= \frac{x^3-4x+4x}{x^2-4}$
$= x + \frac{4x}{x^2-4}$

$y = \frac{x^3}{x^2-4} = x + \frac{4x}{x^2-4}$
$x \neq \pm 2$ $\quad y \neq x$

c) i) $y = \ln(x+1)$ $\quad x+1=1$, $x=0$

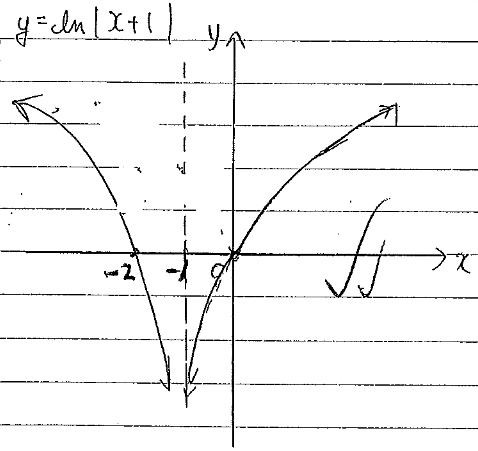


iv) $y = \frac{1}{\ln(x+1)}$

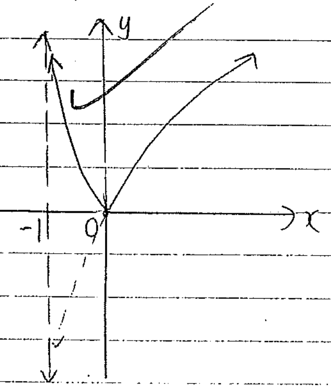


Very close!

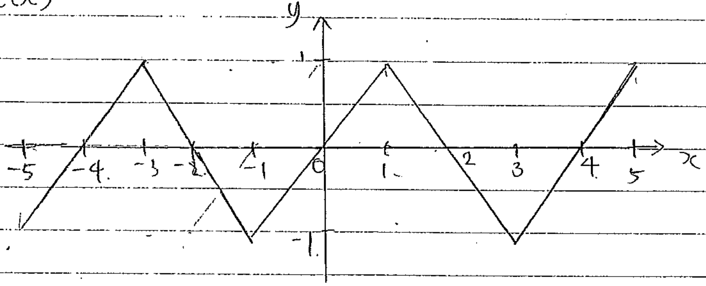
ii) $y = \ln|x+1|$



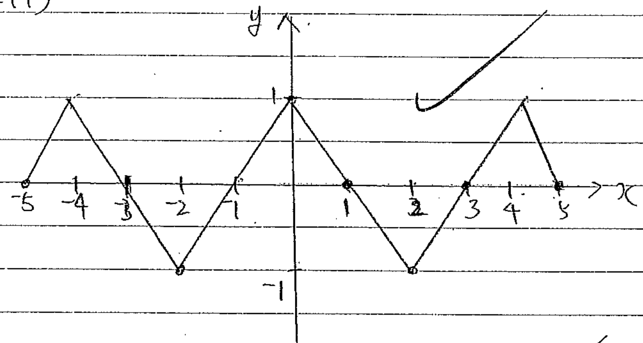
(iii) $y = |\ln(x+1)|$



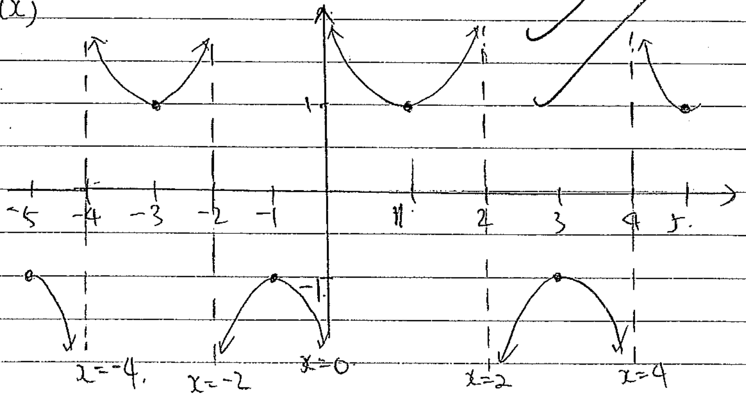
(d) $y = h(x)$



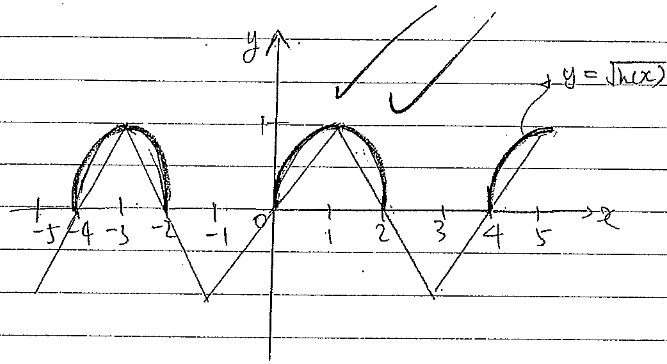
(i) $y = h(x+1)$



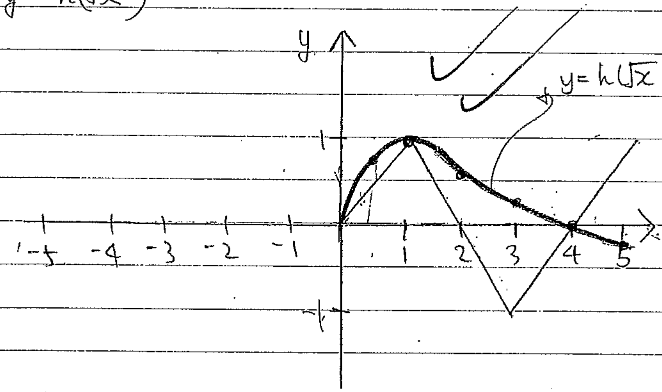
(ii) $y = \frac{1}{h(x)}$



(iv) $y = \sqrt{h(x)}$

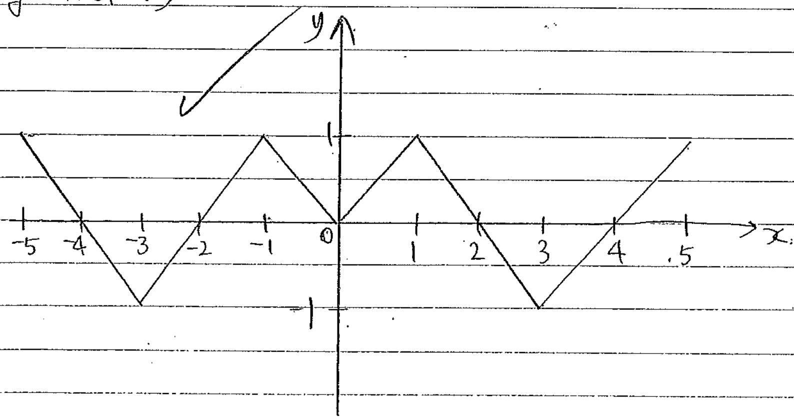


(v) $y = h(\sqrt{x})$



excellent!

(iii) $y = h(|x|)$



Q2.

a) $H: \frac{x^2}{9} - \frac{y^2}{16} = 1$

i) $a^2 = 9, \quad b^2 = 16$

$b^2 = a^2(e^2 - 1)$

$16 = 9(e^2 - 1)$

$e^2 - 1 = \frac{16}{9}$

$e^2 = \frac{25}{9}$

$e = \frac{5}{3} \quad (e > 0)$

$a = 3, \quad S(\pm ae, 0) = (\pm 5, 0)$

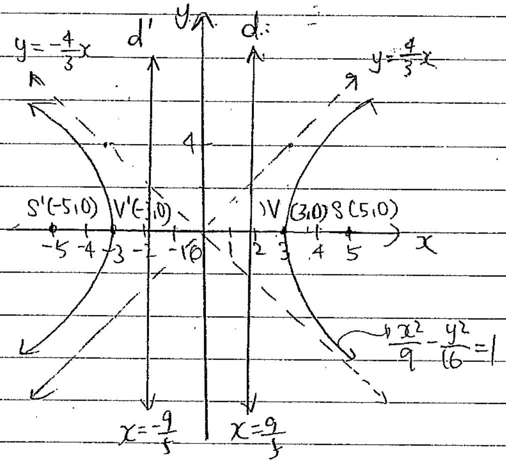
$d: \ x = \pm \frac{a}{e} = \pm \frac{3}{\frac{5}{3}} = \pm \frac{9}{5} \ (\pm 1\frac{4}{5})$

$b = 4$

asymp: $y = \pm \frac{b}{a}x = \pm \frac{4}{3}x$

$V(\pm a, 0) = (\pm 3, 0)$

(ii)



(b) $E: \frac{x^2}{100} + \frac{y^2}{75} = 1$ $\quad a^2 = 100, \ a = 10$; $b^2 = 75, \ b = 5\sqrt{3}$

$b^2 = a^2(1 - e^2)$

$75 = 100(1 - e^2)$

$1 - e^2 = \frac{75}{100}$

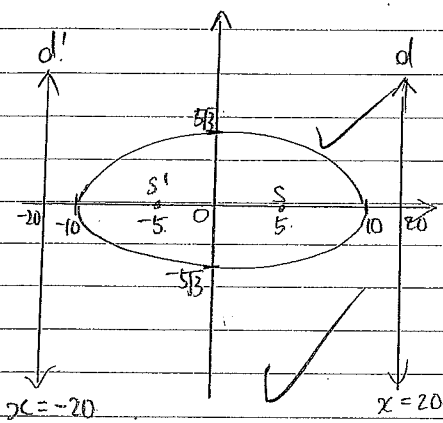
$e^2 = \frac{25}{100}$

$e = \frac{5}{10} \quad (e > 0)$

$S(\pm ae, 0) = (\pm 5, 0)$

$V(\pm a, 0) = (\pm 10, 0)$

$d: \ x = \pm \frac{a}{e} = \pm \frac{10}{\frac{1}{2}} = \pm 20$



(ii) $\frac{x^2}{100} + \frac{y^2}{75} = 1$

$\frac{2x}{100} + \frac{2y}{75} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{100} \cdot \frac{75}{y} = -\frac{3x}{4y}$

At $P(5, \frac{15}{2})$, $\frac{dy}{dx} = -3 \cdot 5 \times \frac{1}{4 \cdot \frac{15}{2}}$

$= -15 \times \frac{1}{30}$

$= -\frac{1}{2}$

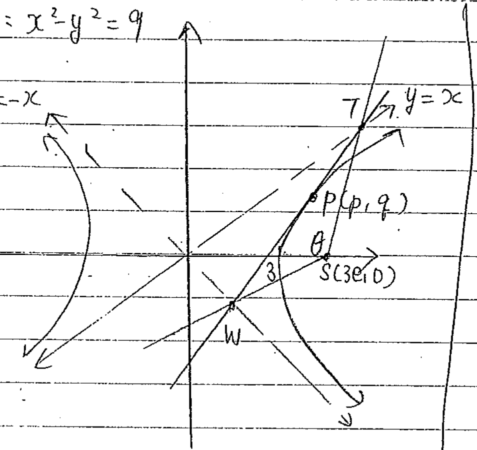
$\therefore$ Slope of normal $= 2$

$N_p: \ y - \frac{15}{2} = 2(x - 5)$

$2y - 15 = 4x - 20$

$\therefore 4x - 2y = 5$ at $P(5, 7\frac{1}{2})$

c) $H: x^2-y^2=9$



(i) $x^2-y^2=9$

$2x-2y\cdot\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{x}{y}$

At P, $\frac{dy}{dx}=\frac{p}{q}$ → slope of tangent (TW)

$\therefore$ TW: $y-q=\frac{p}{q}(x-p)$

$qy-q^2=px-p^2$

$px-qy=p^2-q^2=9$ (P lies on H)

$\therefore px-qy=9$

(ii) $\begin{cases} px-qy=9 \\ y=-x \end{cases}$

$px-q(-x)=9$

$px+qx=9$

$x(p+q)=9$

$x=\frac{9}{p+q}$ $y=-\frac{9}{p+q}$

$\therefore W\left(\frac{9}{p+q}, -\frac{9}{p+q}\right)$

$m_{SW}=\dfrac{\frac{9}{p+q}}{3e-\frac{9}{p+q}}=\dfrac{9}{3e(p+q)-9}=\dfrac{3}{e(p+q)-3}$

(ii) $\begin{cases} px-qy=9 \\ y=x \end{cases}$

$px-qx=9$

$x=\frac{9}{p-q}$, $y=\frac{9}{p-q}$

$\therefore T\left(\frac{9}{p-q}, \frac{9}{p-q}\right)$

$m_{ST}=\dfrac{\frac{\cancel{9}3}{p-q}}{\frac{\cancel{9}3}{p-q}-\cancel{3}e}=\dfrac{3}{3-e(p-q)}$

Let $\angle WST=\theta$

$\tan\theta=\left|\dfrac{m_{SW}-m_{ST}}{1+m_{SW}\cdot m_{ST}}\right|$

$=\left|\dfrac{\frac{3}{e(p+q)-3}-\frac{3}{3-e(p-q)}}{1+\frac{3}{e(p+q)-3}\cdot\frac{3}{3-e(p-q)}}\right|$

$=\left|\dfrac{3(3-e(p-q)-e(p+q)+3)}{(e(p+q)-3)(3-e(p-q))+9}\right|$

$=\left|\dfrac{3(6-e(p-\cancel{q}+p+\cancel{q}))}{3e(p+q)-e^2(p^2-q^2)-\cancel{9}+3e(p-q)+\cancel{9}}\right|$

$=\left|\dfrac{3(6-2ep)}{3e(p+\cancel{q}+p-\cancel{q})-e^2\cdot 9}\right|$

~~(p lies on H)~~

$=\dfrac{18-6ep}{6ep-9e^2}$

$=\dfrac{18-6ep}{6ep-18}$ $(e=\sqrt{2})$

$=-1$

$\therefore \tan\theta=-1$